OFFICE OF FAIR TRADING

Mortgage Redemption Fees

Modelling Redemption Fees and Incentives on UK Home Mortgages and Modelling Variable and Fixed Rate Lending

A research paper prepared
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by Frank Skinner
of Reading University

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MORTGAGE REDEMPTION FEES

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EXECUTIVE SUMMARY

This report develops a theoretically correct methodology for modelling fixed and variable rate mortgages and fixed and variable rate deposit taking. We explain the theoretical basis and justify the particular form of the model we chose. We give simple examples illustrating how the model works. Empirical and practical problems are discussed and solutions are suggested so that the reader can apply the theoretical model to practical problems. The latter feature is especially important for we notice that the results obtained can be sensitive to the quality of parameter estimates.

We show how to use this methodology when mortgages are complex packages of securities that include discounted interest rate periods, cash backs and interest rate caps and floors. We also apply this methodology to modelling the spread between fixed and variable rate deposit taking. The methodology is robust, for we demonstrate how to use it in a practical context using actual examples. Finally, we include a technical appendix describing how these spreadsheets have been constructed and used (A1). We are also making available a copy of the Excel spreadsheet program used to calculate our examples - see page 2 for details of how to obtain a copy.

We find that UK home mortgages and variable and fixed rate deposits are complex packages of securities. We follow a building block approach by first breaking down the mortgage or deposit contract into its component securities, valuing each separately and then adding them up to determine if the full package is priced fairly.

To value each component, we apply a theoretical model of stochastic interest rates that will generate all possible interest rates that may evolve during the life of the contract. We have some assurance that these future rates are correct because we adjust this structure of future interest rates such that they agree with the existing UK Gilt term structure. There should be no discrepancy between the value of a financial asset found by this structure of interest rates and the market price. There should be no discrepancy because if there were it would imply that the UK Gilt market and the value of the financial asset under examination were inconsistent, allowing traders the opportunity to make riskless profit for no investment. This model then values the component security such that the value thereby obtained is consistent with the UK Gilt market. We note that the City uses this approach for valuing financial securities that are in the same spirit and form as the securities that we value in this report, so we expect that practitioners will be inclined to accept results found using this sort of model.

We suggest that redemption fees imposed by UK financial institutions on fixed rate home mortgages can be justified as compensation for implicitly selling a call option to the mortgage holder whenever a fixed rate mortgage is accepted. The value of this implied call option should equal redemption fees, explicit and implicit, plus a modest administrative fee if the redemption fee is to be considered fair.

This comparison ignores the impact of non-financial early repayments, where mortgage holders are forced by circumstances beyond their control to re-finance at higher (rather than lower) mortgage rates. If financial institutions have a blanket policy of always charging the full redemption fee, even when re-borrowing occurs at higher rates, the above method of determining whether redemption fees are fair would be biased in favour of financial institutions. If that is the case, then the value of the implied option should be slightly lower than suggested by our methodology.

We value variable rate mortgages that include cash backs and discounted mortgage rate incentives. We note that these incentives often include implicit securities that place the mortgage holder under obligation and are valuable to the financial institution. Sometimes the value of these securities represents a large fraction of the stated value of the “incentive”. We speculate that most mortgage holders are financially unsophisticated and are vulnerable to being misled as to the true value of the incentive. We suggest that rather than restricting consumer choice by prohibiting such terms, the financial institution should be required to reveal the cost of these additional obligations early in the mortgage negotiation process. In other words, if we allow for consumer choice, let it be an informed choice.

Finally, we model the spread between instant access and time deposit rates. While we are unable to determine whether the level of these rates is fair, we are able to determine whether the spread between them is fair. We find that the spread is in part determined by an implicit put option forgone by a deposit holder choosing to lend at fixed rather than variable term rates.

1.0 INTRODUCTION

This report develops a theoretically correct methodology for modelling fixed and variable rate mortgages and the associated redemption fees. We show how to use this methodology when these mortgages are complex packages of securities that include discounted interest rate periods, cash backs and interest rate caps and floors. The methodology is robust, for we demonstrate how to use it in a practical context using actual examples. We also apply this methodology to modelling the spread between fixed and variable rate deposit taking.

The methodology we develop is based on the arbitrage free approach pioneered by Ho and Lee (1986). We take care in explaining the theoretical justification of this approach. We note that the City uses this approach for valuing financial securities that are in the same spirit and form as the securities that we value in this report, so we expect that practitioners will be inclined to accept results found using this sort of model. We discuss several variations of this model and choose one that is immediately applicable to the tasks at hand. Then we apply this methodology to the tasks outlined above.

1.1 General Approach and Organisation of the Report

The basic focus of this report is to develop a methodology that will determine whether a mortgage offer and its associated redemption fee are “fair”. The spread between variable and fixed rate-lending rates is also considered. Mortgage offers in the UK are complex financial contracts that may include several implicit securities. We follow a building block approach by first breaking down the mortgage offer into its’ component securities, valuing each separately and then adding them up to determine the value of the full package. One can then judge whether the mortgage offer is “fair” by comparing the theoretically correct value to the stated value of the mortgage offer. We use this approach to value fixed and variable rate mortgages. Importantly, we value additional mortgage contract terms that upon careful examination reveal themselves to be options. These additional contract items are redemption fees, and interest rate floors and caps.

Redemption fees compensate the financial institution for a call option they implicitly sell to the mortgage holder whenever the financial institution offers a fixed rate mortgage. To see this, recognise that the fixed rate mortgage holder has the right but not the obligation to refinance at any time prior to the end of the agreed term. This option is valuable, for if mortgage rates decline, the mortgage holder can redeem the mortgage prior to the end of the agreed term and refinance at a lower rate thereby reducing his/her monthly mortgage payment. This gain comes at the expense of the financial institution, which now receives a lower mortgage payment for the same amount borrowed.

On the other hand, if mortgage rates increase, the mortgage holder will *not normally* redeem the mortgage early since this would imply replacing a low cost mortgage with a high cost mortgage. So the financial institution cannot usually hope to recover part of the cost of providing the implied option through unfavourable (from the mortgage holder’s perspective) mortgage refinancing. Hence a redemption fee must be charged to recover the value of the option implicitly sold to fixed rate mortgage holders. To be a “fair” charge we would expect this redemption fee to equal the value of the option implicitly sold plus a modest fee to recover the incremental costs incurred in administering the redemption.

Two additional points should be made at the outset. First, the above reasoning suggests that there should be only a small redemption fee on variable rate mortgages. Variable rate mortgages reset the variable rate to the current rate periodically. In effect, the financial institution agrees to automatically refinance the mortgage at the going rate, sometimes to the mortgage holder’s advantage (when mortgage rates decrease) sometimes to the mortgage holder’s disadvantage (when mortgage rates increase). Hence, neither party has an option, both agree to pay/receive the going mortgage rate. Since the financial institution does not need compensation for selling an option, the redemption fee should reflect an administrative fee to cover the administrative cost imposed upon them by the process of refinancing.

Second, while it is conventional to think that mortgage holders will not refinance if mortgage rates increase, we can expect that some will. Mortgage holders sometimes refinance for non-financial reasons. Some repay mortgages early because of marital break-ups, death, unemployment or, more optimistically, because of career opportunities in other areas of the country. A financial institution with a large portfolio of mortgages can reasonably anticipate a portion of their mortgages will be redeemed early for non-financial reasons. This implies that some mortgage refinancing will occur at unfavourable rates (from the mortgage holder’s perspective), implying partial cost recovery of the implied option sold by the bank to fixed rate mortgage holders in general.

This report will ignore the possible effect of non-financial refinancing under the assumption that the financial institution waives most of the redemption charge (charging only for the administrative cost of redemption) in the event of a non-financial re-mortgage at unfavourable mortgage rates. If this assumption is not valid, we expect that the true cost of the call option implicitly given to the fixed rate mortgage holder will be somewhat less than the value suggested by this report’s methodology.

Therefore one of the challenges addressed in this report is to value the option implicitly sold to fixed rate mortgage holders whenever a financial institution offers them a fixed rate mortgage. This option is American since the mortgage holder may exercise their option at any time until the end of the fixed rate period. Since the value of the option is driven by stochastic interest rates, we must use an option-pricing model that treats interest rate variability in an acceptable way.

Interest rate floors and caps are often included in variable rate incentive mortgages. A floor represents the obligation of the variable rate mortgage holder to continue paying a fixed floored rate when the unrestricted variable rate is lower. If the unrestricted variable rate is above the floored rate, the variable rate mortgage holder pays the higher unrestricted variable rate. In this report we argue that this obligation represents a put option sold by the mortgage holder. Meanwhile, a cap represents an obligation by the financial institution to accept the capped mortgage rate when the unrestricted variable rate is higher. If the unrestricted variable rate is below the capped rate, then the financial institution will accept the lower unrestricted variable rate. In this report we argue that this obligation represents a call option sold by the financial institution.

We note that a large fraction of the value of an "incentive" can actually represent the value of a put option (floor) sold by the mortgage holder. Hence the stated value of the incentive may be misleading. The stated value implies that this is the amount the financial institution is willing to pay to obtain the mortgage holder's business when in fact a large fraction of the incentive can actually represent a payment to the mortgage holder for accepting an obligation. We speculate that the mortgage holder is financially unsophisticated and so is vulnerable to being misled as to the value of an incentive mortgage offer that includes an interest rate floor.

This report is organised as follows. In the next section we review the arbitrage free approach and justify the particular model chosen for this study. In Section 3, we consider the empirical issues that inevitably arise in empirical studies. Section 4 considers practical issues that arise when a theoretical model is applied to a practical problem. As an illustration we consider how to model redemption fees in the UK home loan market. Section 5 discusses how we intend to value redemption features. We carry out actual numerical examples in Sections 6 and 7 to demonstrate how one can apply the model bearing in mind our solutions to empirical and practical issues previously discussed in Sections 3 and 4. Section 6 considers how to value a redemption fee on a fixed rate mortgage. Section 7 considers how to value variable rate mortgages. In this section we include two actual examples of variable rate mortgages that include incentives, one a cash back and the other a discount mortgage. Both examples include interest rate floors. Section 8 models the spread between fixed and variable rate deposit taking. We use two examples - first, modelling the spread between instant access rates and a 60-day time deposit rate and, second, the spread between instant access rates and a 90-day time deposit rate. Section 9 discusses the policy implications of this report. A technical appendix (A1) describes the basic Excel spreadsheet, how it is constructed and used. Copies of the main spreadsheet program used in the report are available from the Office of Fair Trading. See page 2 for further details.

2.0 ARBITRAGE FREE MODELLING OF STOCHASTIC INTEREST RATES

It is all too easy to get caught up in the details of a theoretical model and miss the overall idea that drives it. Yet it is vitally important to understand the basic idea behind a model to appreciate its power and validity. So first we will explain the basic intuition behind arbitrage free pricing. Then we will go through the details of the model and conclude by summarising how the two fit together.

2.1 Arbitrage Free Pricing

The general approach used by interest rate arbitrage free pricing is replicating observed market prices of zero coupon Treasury securities through investing in portfolios of Arrow-Debreu securities. These Arrow-Debreu securities are then used to price interest rate options. Since these Arrow-Debreu securities replicate observed Treasury market prices, then the interest rate option must be the value obtained by the structure of Arrow-Debreu prices. Otherwise the cash bond and bond option markets would be inconsistent. This will allow for arbitrage, the ability to earn money through no investment or risk taking. In other words, earn money for nothing[1]. The incentive to gain wealth for no effort is so powerful that we believe this kind of opportunity cannot last for long. Traders will attempt to capture this free money, and in the process bid up the price of undervalued, and bid down the price of overvalued securities till the cash bond and option markets agree with one another. Since competitive markets do not allow arbitrage opportunities to exist, then the structure of arbitrage free Arrow-Debreu prices can be used to accurately price interest rate options. Hence, this modelling approach is termed arbitrage free pricing.

And now for the details. First, what is an Arrow Debreu security? It is a hypothetical security that will pay £1 next period if a particular interest rate level occurs, zero otherwise. We say we have a complete market if we have an Arrow-Debreu security for each possible interest rate level not only next period, but in all future periods. If we have a complete market, then we can buy a portfolio of Arrow-Debreu securities such that next period (or any other period for that matter), we are guaranteed the payment of £1 next period, no matter what interest rate level actually evolves. This is exactly what a zero coupon Treasury bond pays. Since the portfolio of Arrow-Debreu securities replicates exactly the same cash flow structure of a zero coupon Treasury bond, then the Arrow-Debreu portfolio (replicating portfolio) and Treasury zero must have the same price. If not, we can conduct pure arbitrage. In other words we gain wealth but do not use any of our own cash and take no risk.

To see exactly how pure arbitrage works, consider the case where the Arrow-Debreu replicating portfolio price is lower than the Treasury price even though both securities pay the same amount (£1) next year. Then we would immediately short sell the Treasury zero. With the money thereby obtained, we buy the replicating portfolio of Arrow-Debreu securities. By construction, the cost of buying the replicating portfolio is less than the proceeds from shorting the Treasury zero. This means that rather than

[1] Later applications will make it clear that the no arbitrage condition is the finance equivalent to ruling out violations in the law of one price.

investing some of our money, we have been paid to short the zero and buy the replicating portfolio. Since we have been paid, we normally expect that we have taken some risk. Next period we must pay £1 to cover our short Treasury zero. However, we obtain £1 from our replicating portfolio. These two positions (short Treasury, long replicating portfolio) *always* offset, so we have no net future obligation, and no risk. Thus we have been paid money for no risk. When we spot such an opportunity, we would immediately buy Arrow-Debreu securities, driving up its price, and short Treasury zeros, driving down its price, until the cost of the replicating portfolio equals the value of the Treasury zero. Therefore, if we believe that no pure arbitrage can exist in a competitive market, then the structure of Arrow-Debreu prices must agree with the Treasury zero price.

The same argument holds for all time periods, the price of multi-period Arrow-Debreu securities that we use to form multi-period replicating portfolios must agree with multi-period Treasury zero prices in order to avoid pure arbitrage in a multi-period context. Furthermore, if Arrow-Debreu security prices are consistent with Treasury zero prices, then *any* security that is dependent upon the value of Treasury zeros must be consistent with replicating portfolios of Arrow-Debreu securities if pure arbitrage is to be disallowed. In other words, we have a general pricing approach (arbitrage fee pricing) that we can use to price any Treasury bond, or option and futures contract.

2.2 Obtaining Arrow-Debreu Prices

Obviously obtaining the correct Arrow-Debreu price is a critical step. How is this to be accomplished? To explain how we do this we will first explore binomial stochastic interest rate processes. From this interest rate process we will generate stochastic discount factors and Arrow-Debreu prices. Finally, we will explain how the interest rate process, discount factors and Arrow-Debreu prices are forced to agree with the Treasury zero coupon term structure.

2.21 The Binomial Interest Rate Process

We start by observing that interest rates are stochastic - they randomly change from period to period. Ho and Lee (1986) suggested that we can represent the process that generates all possible future rates of interest as a binomial process. Their process was,

$$R_{t,i} = R_{0,0} + \sum_{t}^{N} u_t T + \sigma\sqrt{T}$$

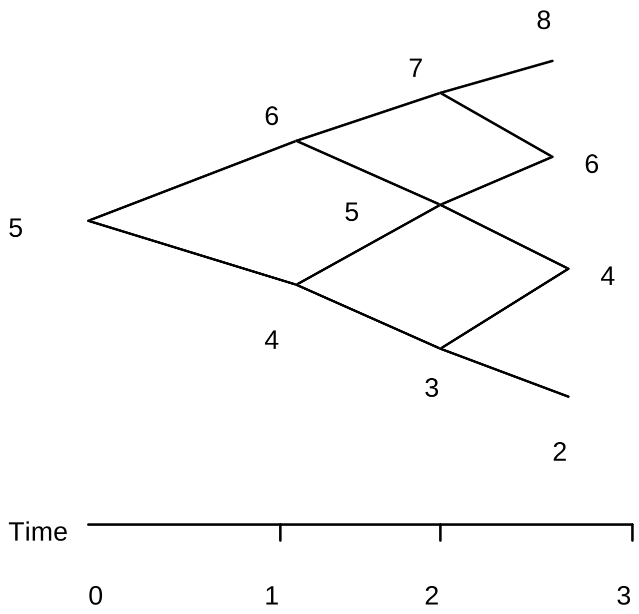
in the up interest rate state and

$$R_{t,i} = R_{0,0} + \sum_{t}^{N} u_t T - \sigma\sqrt{T}$$

in the down interest rate state, where $R_{t,i}$ = the future one period interest rate that evolves at time t and interest rate state (level) i, $R_{0,0}$ is today's one period risk free rate of interest, u_t is a calibration factor that we use to adjust the above interest rate process to the existing zero coupon Treasury yield curve, T is the time step, usually one month for home mortgages, and σ is the standard deviation (volatility) of the interest rate process.

No doubt a numerical example will help us to understand the process. Suppose the six-month Treasury zero (T-bill) interest rate is 5%, the time step is one year and the volatility of interest rates is 1%. For the moment, we will ignore the calibration factor. Then future rates of interest will evolve as in Figure 1.

Figure 1-Interest Rate Tree



This diagram shows that with probability 0.5, interest rates may jump up by 1% from 5% to 6%, or alternatively, interest rates may decrease by 1% to 4% with probability 0.5. In the second period, if the up branch of the above tree has been taken, then interest rates may decrease (with probability 0.5) from 6% to 5%, or it may increase (again with probability 0.5) from 6% to 7%. Alternatively, if the down branch of the above tree has been taken, then interest rates may decrease from 4% to 3% or it may increase from 4% to 5% all with probability 0.5. The above interest rate tree represents all possible paths the short rate of interest may follow in three years. The idea is that long rates of interest are nothing more than a sequence of short rates. Accordingly, we can view an interest rate tree as a sequence of all possible term structures as they may occur in the future.

A few comments are in order at this point. Most will react in horror at the apparent oversimplification of the interest rate process. First, recognise that we are using a time step of one year for illustrative purposes only. It is much more realistic to use a shorter time interval, say one month, so that the above jumps in interest rates will be much smaller. Second, we will use many more time steps than three to value the redemption feature, typically 60. Since the above process is binomial, the sequence of the above binomial interest rates will converge to a normal distribution. In other words, we are assuming that interest rates are normally distributed. Since this still allows for unrealistic negative interest rates (just think of the lowest branch of the tree at the sixth time step) we will adjust the above Ho and Lee (1986) process according to Black Derman and Toy (1990). This process is very similar to Ho and Lee (1986) interest rate process, but it is log-normally distributed, so it prevents negative interest rates. In other words, the above interest rate process can be easily adjusted to generate a reasonable interest rate tree that we can claim to represent all possible interest rates that may evolve in the future.

This interest rate process is

$$R_{t,i} = R_{0,0}.\exp\left[\sum_{t}^{N} u_t T + \sigma\sqrt{T}\right]$$

in the up interest state and

$$R_{t,i} = R_{0,0}.\exp\left[\sum_{t}^{N} u_t T - \sigma\sqrt{T}\right]$$

in the down interest rate state. Notice all we have done is *multiply* the *exp* of the term in square brackets by the current short-term interest rate rather than *add* the term in square brackets to the current short rate of interest as was done in Ho and Lee (1986). This simple adjustment prevents the above interest rate process from generating negative interest rates and converts the normal distribution of interest rates into a log-normal distribution. This adjustment was first suggested by Black Derman and Toy (1990), and is often called Black Derman and Toy, constant volatility version. This is the process we will use in this report.

2.22 Discount Factors

The above interest rate process generates an interest rate tree that represents all possible future short rates of interest. We now find one period discount factors as follows.

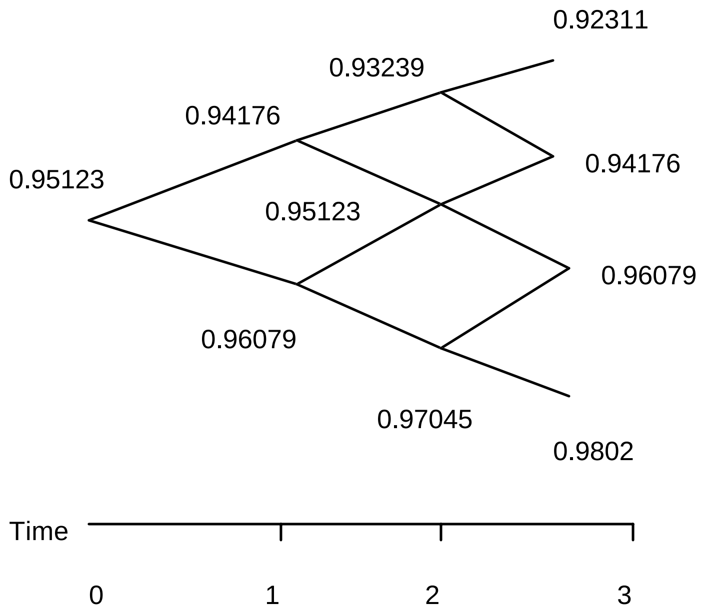
$\text{Exp}^{-(R_{t,i}T)}$.

Where $R_{t,i}$ is the interest rate that evolves at time t and interest rate state (level) i and T is the time step of one month (1/12=0.08333 of a year). This discount factor will discount the value of £1 received at the end of the month to its value at the beginning of the month assuming continuous compounding. The assumption of continuous compounding is innocuous since we find all values assuming continuous compounding so all values are treated equally. We must use continuous compounding since our interest rate process is generated by a log-normal distribution which itself is a continuous process.

Hence we will find a tree of one period discount factors. Each node of the tree represents the beginning of period value of receipt of £1 at the end of the period. In this way we can find the present value of any cash flow by rolling backwards through the tree. Exactly what this means is demonstrated by a numerical example.

Using Figure 1, we generate the discount factor tree reported in Figure 2. Notice that assuming a time step T of 1 and continuous discounting generates these discount factors. For example, at state S(2,2), which represents the second time period and two interest rate levels above the lowest possible interest rate level at t=2, the discount factor is £1*$\exp^{-(0.07)x1}$ = 0.93239.

Figure 2-Discount Factor Tree



Suppose we wish to value a four period Treasury zero that will pay £100 at the beginning of the fourth period. Straight away we see that its value will be £92.311 in state S (3,3) since £100 *0.92311 = £92.311. Notice that S (3,3)'s value is the numerical representation of S (t,i) where t=3 (third period) and i =3 (three interest rate levels up from the lowest interest rate level of 2%). It is simpler to represent the value of this Treasury zero in state S (3,3) as Z (3,3) = £92.311. Similarly at Z (3,2), the Treasury zero is worth £94.176, Z (3,1) = £96.079 and finally Z (3,0) = £98.02.

At two periods prior to maturity, we find the value of the zero at state S (2,2) as

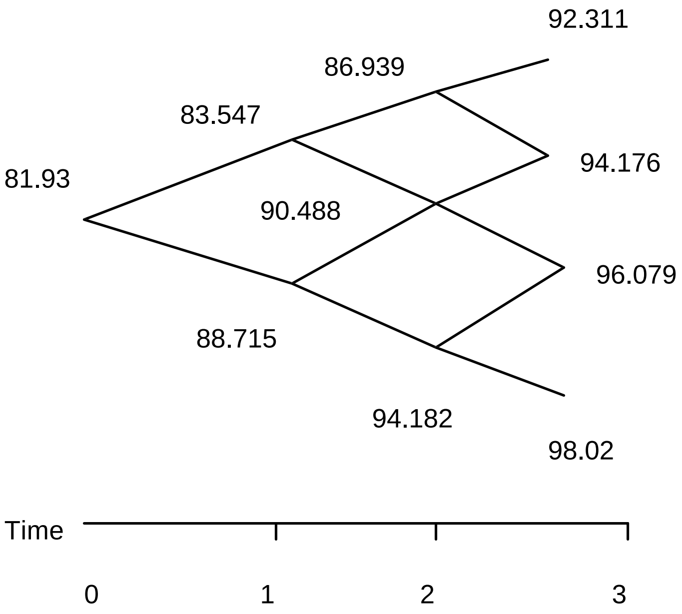
Z (2,2) = 0.5[Z(3,3)+Z(3,2)]*d(2,2) = 0.5[92.311+94.176](0.93239)
= 86.939

Notice that d (2,2) refers to the discount factor at state S (2,2). Similarly, at S (2,1)

Z (2,1) = 0.5[94.176+96.079](0.95123) = 90.488

And so on. Continuing this process Figure 3 shows that we find the value of the four-period zero as £81.93 and its yield is approximately 5% on an annual compound basis (–ln [(Z (0,0)/100])/4.

Figure 3-Zero Coupon Bond Price Tree



2.23 Arrow-Debreu Prices

We now have all the information necessary to find the Arrow-Debreu prices. Suppose you wish to buy a security that will pay £1 next period only if interest rates rise from 5% to 6% next period, zero otherwise. What is this security worth? We know the value of a security that will pay £1 next period no matter if interest rates rise or fall, for this type of security is a Treasury zero. Per £100 of face value it is worth £100*0.95123 (see d (0,0) in Figure 2) = £95.123, so per £ of face value the Treasury zero is worth 95.123p. But the Arrow-Debreu security will pay £1 half of the time next period since interest rates are equally likely to rise or fall. So we can say its value is 0.5*d(0,0) or 0.5*(95.123p) = 47.5615p. This makes sense because if you buy two Arrow-Debreu securities, one paying off only if interest rates rise, and one paying off only if interest rates decrease you will have bought a synthetic Treasury zero, the value of which is 47.5615*2=95.123p. A synthetic zero that does exactly the same as a real zero must have the same price to avoid pure arbitrage (avoid violations in the law of one price).

From this discussion we see that the value of a one period Arrow-Debreu security is

$$A(1,i) = 0.5 * d(0,0)$$

A (1,i) is the value of the Arrow-Debreu security that pays off £1 at time 1 in state i and d(0,0) is the one period discount factor as of the present time. Note that the discount factor adjusts for time value, and our probability of 0.5 adjusts for interest rate risk.

What about Arrow-Debreu securities that will pay off £1 in two periods' time only if a particular interest rate state level occurs? The added complication is that for the middle interest rate states there is more than one way this interest rate level may occur. For example, in period two we may obtain a 5% interest rate level by experiencing a decrease from 5% to 4% in the first period and then experiencing an increase from 4% to 5% in the second. Similarly, we may also get to 5% in the second period because we may have experienced an "uptick" in interest rates in the first period, followed by a "downtick" in interest rates in the second period. As you may well expect, we simply add together the value of both possibilities. That is,

0.5*d(0,0) *0.0.5*d(1,1) + 0.5*d(0,0) *0.5*d(1,0)

= 0.5*(.95123)*0.5(.94176) + 0.5*(.95123)*0.5*(.96079)

= 0.22396 + 0.22848 = 0.45244

Notice that multiplying d(1,1) by d(0,0) traces out the time value of money of the certain receipt of £1 in two periods' time if interest rates rise at the beginning of period one. Similarly, multiplying d(1,0) by d(0,0) traces out the time value of money of the certain receipt of £1 in two periods' time if interest rates fall at the beginning of period one. Therefore the discount factors adjust Arrow-Debreu security prices for time value.

In the first instance, payment of £1 only occurs if interest rates rise from 5% to 6% at the beginning of period one, and then fall from 6% to 5% at the beginning of period two. Each interest rate movement occurs with probability of 0.5, so in total this sequence of interest rate movements will occur with a probability of 0.5*0.5 or 0.25. Similarly, a down and up sequence in interest rates occurs with probability 0.25. Therefore these probabilities adjust Arrow-Debreu security prices for interest rate risk. For the two extreme interest rate paths at time period two, two upticks in interest rates and two downticks in interest rates, there is only one path.

In summary Arrow-Debreu security prices represent the present value of uncertain receipt of £1 in the future. These prices are adjusted for interest rate risk and time value.

There is a convenient representation of the above pricing procedure called forward induction. The result can be stated as follows,

$$A(t,i) = 0.5.[A(t-1,i).d(t-1,i) + A(t-1,i-1).d(t-1,i-1)]$$

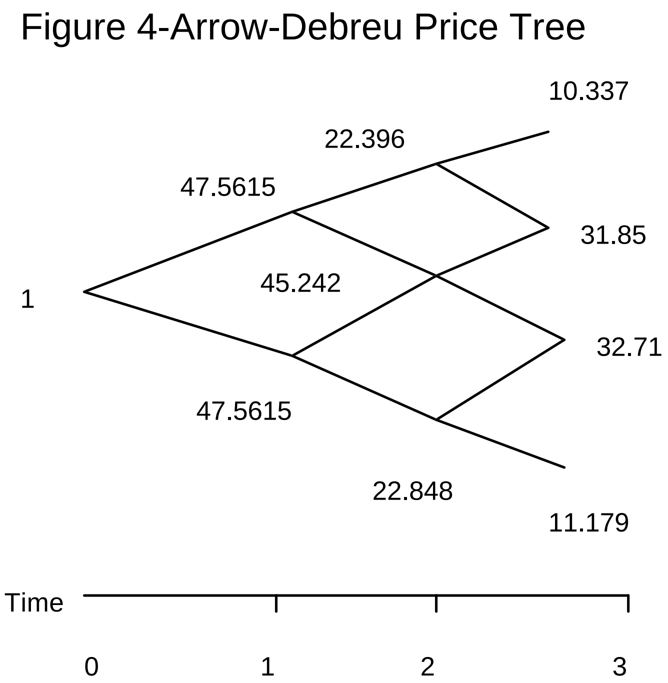
where for all i-1< 0, A(t-1,i-1) = 0. For example, a security that pays off £1 in the second period only if interest rates fall twice is,

$$A(2,0) = 0.5[A(1,0).d(1,0) + A(1,-1).d(1,-1)] = 0.5[0.475615 * 0.96079 + 0] = 0.22848$$

or 22.848p. For an Arrow-Debreu security that pays off £1 if the middle interest rate level is reached in period two the price is,

$$A(2,1) = 0.5[A(1,1)d(1,1) + A(1,0)d(1,0)]$$
$$= 0.5[0.475615 * 0.94176 + 0.475615 * 0.96079]$$
=0.45244.

Notice that this procedure is called forward induction as it allows us to calculate prices of Arrow-Debreu securities period by period based on information generated at the beginning of each period. Figure 4 shows the Arrow-Debreu price tree that is built using the discount factors that in turn is generated by our interest rate process.



2.24 Calibration of the Interest Rate Process to the Existing Term Structure

The above Arrow-Debreu security tree allows us to calibrate the interest rate process to the existing sovereign term structure. By doing so, we incorporate market information, in particular, market risk aversion, into our interest rate process. Additionally, this calibration forces our structure of Arrow-Debreu security prices to agree with the UK debt market, so any security we value using these prices must be correct. Otherwise the security we price is inconsistent with the UK sovereign debt market and we can conduct pure arbitrage.

To do this, notice that if we buy one each of A(3,3), A(3,2), A(3,1) and A(3,0), we would have a portfolio of securities that will pay £1 at the end of period three, no matter what interest rate state evolves at the beginning of period three. This is exactly what a Treasury zero does, so the price of this Arrow-Debreu portfolio must be the same as the price of a three period Treasury zero.

In this case, the value of this portfolio is 86.076p. The continuously compounded total return is –ln (86.076/100) = 15%. The annually compounded equivalent is simply this value divided by (1/m)xn where m is the number of compounds per year and n is the number of years. Since we have used a time step of one year, m=1 and we have three years so the annually compound yield is 5%.

Now we check to see if the UK Treasury yield is 5%. If not, we adjust the parameter u_t in our interest rate process. As we adjust u_t the interest rate tree changes. In turn this changes the discount factor tree and then the Arrow-Debreu tree. We continue to do this until the structure of A(3,3)+A(3,2)+A(3,1)+ A(3,0) obtains a price whose annual yield agrees with the UK annually compounded three year Treasury yield.

2.25 Alternative Models

Within the arbitrage free class of pure interest (credit risk free) interest rate models, there are a number of alternative models. Each of these alternative models differs in the interest rate process they use. In particular, Black Derman and Toy (1990) (stochastic volatility version), Hull and White (1990) and Heath, Jarrow and Morton (1992) are the relevant alternatives. Additionally, Jarrow and Turnbull (1995) adjust the above interest rate models for credit risk.

In general, while the alternative pure interest models offer potential improvements, typically these improvements are marginal. On the other hand, each of these alternative interest rate processes requires estimates of additional parameters that can introduce additional error. For example, Black Derman and Toy (1990) stochastic volatility version assumes volatility is not constant, but evolves as a binomial tree. This volatility tree is calibrated to an existing volatility curve in the same way our interest rate process is calibrated to the existing UK Treasury yield curve. However the improvements this model offers depend upon an accurate estimate of the existing volatility curve, so adding more complexity also adds additional sources of error.

Similarly, Hull and White (1990) model the interest rate process as a trinomial rather than a binomial process. This means that at any particular point in time, the interest rate may move to three new levels next period, not necessarily up, down and stay the same. This additional flexibility comes at a cost; one must not only obtain knowledge of the volatility structure, but also the rate of mean reversion. The latter refers to the rate at which abnormal interest rates revert to some central tendency interest rate.

Finally, Heath, Jarrow and Morton (1992) use the forward curve rather than the spot term structure. This model requires calibration of both the forward curve and the volatility of forward rates. Unfortunately, this generates a "non-recombining tree", meaning that in two periods' time, an interest rate path traced out by an uptick followed by a downtick in interest rates does not lead to the same interest level as a downtick followed by an uptick. This generates a large number of possible interest rate levels within a few dozen time steps and makes the model extremely slow to run. To estimate the model, one must use Monte Carlo simulation, which cannot value American options. We argue below that the right to redeem a fixed rate mortgage early is an American option, so the Heath Jarrow and Morton (1992) approach is not appropriate for the task at hand.

Finally, Jarrow and Turnbull (1995) adjust any arbitrage free model for credit risk. Again this model requires additional information. In particular, one must know the recovery rate, the fraction of promised cash flows paid back in the event of bankruptcy. Nevertheless this model adjusts values for credit risk which at first glance might be considered critical. However, recall that we are valuing mortgages and options on mortgages that are bought by high credit quality financial institutions, and are sold to mortgage holders who are also of high credit quality. Furthermore, a highly marketable asset (a house) secures the loan where the loan amount is typically far less than 100% of the market value of the asset. Since credit risk is low, and probably at the same level on both sides of the mortgage transaction, ignoring credit risk considerations probably makes no difference. It is not worth our while to introduce additional error through a guess of the recovery fraction to adjust for a relatively minor differential factor, credit risk.

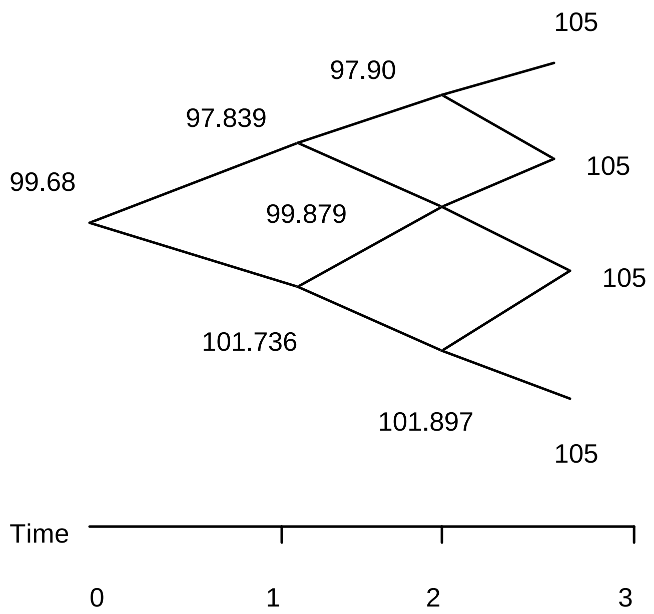
In any event, practitioners have developed proprietary versions of these stochastic interest rate models in the hope that with more complex interest rate processes they can obtain more accurate pricing to gain greater profits from trading. This knowledge-based competition implies they will be reluctant to tell anybody exactly what model they use. Hence there is no hope that we can guess exactly what version of the above arbitrage free model they use. Nevertheless, our parsimonious model should be able to closely replicate their results since our model is in the same class of models as theirs. We are able to see clearly if items such as redemption fees are priced fairly without the added complexity and potential error of marginally improved interest rate models.

2.26 Summary

We have seen that we can develop an interest rate process that assumes that interest rates are stochastic. The suggested process is realistic since within a few time steps, a large distribution of possible non-negative interest rates can be developed. This interest rate tree is used to generate an Arrow-Debreu price tree. By calibration, we force our interest rate process, discount factor tree and Arrow-Debreu security tree to agree with the UK Treasury bond market.

Because the Arrow-Debreu *and* discount factor trees are calibrated to the UK Treasury debt market, then prices of any security depended upon the UK Treasury interest rates must agree with prices obtained by *either* using the Arrow-Debreu prices *or* the discount factors. Usually it is more convenient to price interest rate sensitive options by rolling back through the tree as demonstrated in Section 2.22. If this price found by the discount factor tree is different from the market price, then an investor can conduct pure arbitrage by buying cheap and selling dear. For example, suppose we find three-year annual pay 5% coupon bonds priced in the market at £100. Using our discount factor tree as shown in Figure 2 we know the price should be £99.68. The calculation is shown in Figure 5.

Figure 5-Coupon Bond Pricing



This value is found by using the 50/50 probability rule and rolling backwards through the discount factor tree as illustrated below.

$$B(t,i) = (0.5[B(t+1,i+1) + B(t+1,i)] + C)d(t,i)$$

Where B (t,i) represents the value of a coupon bond at time t and interest rate level i and C is the coupon payment. For example,

B (1,1) = (0.5[97.901+99.879]+5) x 0.94176=97.839

We can then buy a portfolio of Arrow-Debreu securities, 105 A (3,3), 105 A (3,2), 105 A (3,1) 105 A (3,0), 5 A (2,2), 5 A (2,1), 5 A (2,0), 5 A (1,1) and 5 A (1,0). Notice that we buy a number of Arrow-Debreu securities that is consistent with the amount and timing of the cash flows from the 5% annual pay coupon bond with a face value of £100. The value of this portfolio is the same as the value given by the discount factor tree (except for a 2p rounding error) since both the discount factor tree and Arrow-Debreu tree are calibrated to the existing Treasury yield curve. This Arrow-Debreu price calculation is given in Table 1.

Table 1
Arrow-Debreu Pricing

Interest Rate State	Promised Cash Flow (1)	Arrow-Debreu Price (2)	Value (1x2)
S(3,3)	105	10.337p	£10.85
S(3,2)	105	31.85p	£33.44
S(3,1)	105	32.71p	£34.35
S(3,0)	105	11.179p	£11.74
S(2,2)	5	22.396p	£1.12
S(2,1)	5	45.242p	£2.26
S(2,0)	5	22.848p	£1.14
S(1,1)	5	47.5615p	£2.38
S(1,0)	5	47.5615p	£2.38
		Total	£99.66

Notice that since an Arrow-Debreu security represents the present value of £1 paid at a particular time and state, it is easy to find the value of an Arrow-Debreu portfolio. It is the sum of products of the number of Arrow-Debreu securities by the corresponding per £ Arrow-Debreu price.

This means we can buy the Arrow-Debreu portfolio at a cost of £99.68, and short the 5% coupon bond at £100. The proceeds from short sale covers the cost of the Arrow-Debreu portfolio and leaves us with £0.32. This is free money, because we can always cover our obligations on the short bond since we are guaranteed that our Arrow-Debreu portfolio will pay-off 5% at t=1 and at t=2, and will pay-off 105 at t=3. Obviously traders will bid up the price of the Arrow-Debreu securities and bid down the price of the coupon bond till both securities have the same price. This is simply a numerical example of the arbitrage free argument introduced at the beginning of this section.

3.0 EMPIRICAL ISSUES

To implement the Black Derman and Toy (1990) model (constant volatility version) we must confront two major issues. First, we need to consider how to estimate the existing Treasury yield curve. Second, we need an accurate estimate of the interest rate volatility parameter. We will discuss each issue in turn in the following two subsections.

3.1 Yield Curve Issues

This issue is critical since the accuracy of the discount factor and Arrow-Debreu price trees depends upon calibration to the correct yield curve. To obtain a useable, accurate yield curve, we must consider three problems.

<u>First, we need a zero coupon yield curve to avoid the coupon effect.</u> The coupon effect is present in the yields of coupon bonds because the yield to maturity of a coupon bond actually represents a weighted average of true yields. This happens because when we calculate the yield to maturity on a coupon bond, all cash flows are discounted at the same yield to equal the current price. Since earlier coupons are paid at earlier times these coupons should be discounted at some other yield. The only case where this is not a problem is when the yield curve is flat (same yield at all maturities) which in general is not the case. Since the yield to maturity calculation ignores the shape of the yield curve, yet uses the correct market determined price, it finds the average yield to maturity that successfully discounts all cash flows to equal price.

The problem is that we wish to value future cash flows using a discount rate (yield) that corresponds to its cash flow structure and time horizon of receipt in order to measure accurately its value. In general, we cannot expect the timing and levels of the cash flows we wish to value to correspond to the cash flow structure and timing of the coupon bond. In other words, the timing and amounts of cash flows from the coupon bond and cash flow we wish to value require different average discount rates.

To avoid this problem we need to use yields on zero coupon bonds. In contrast to coupon bonds, zeros return cash only at the maturity date, so its yield corresponds to the yield on cash invested for the full maturity. When we need to value a cash flow structure, we simply discount each cash flow at each maturity date using the corresponding maturity zero yield.

<u>Second, we need accurate bond prices in order to estimate accurate zero yields</u>. The problem is that many bonds do not trade very often, so many publicly available prices are actually indicator prices. These represent somebody's best guess, which may not be very accurate. To avoid this problem, we use only those bond prices where the underlying bond issue is £50 million or more. The idea is that the larger the issue, the more likely it is that some of the bonds have been traded recently. Furthermore, the larger the issue, the more traders would be dealing in them. This means that we can expect more accurate indicator prices even when the bond has not been traded recently because there would be many traders familiar with the issue willing to give quotes.

Third, we need a *continuous* yield curve since we would like to price redemption features written on mortgages where payments on the mortgage may occur on any day of the month, and these mortgages can run as long as thirty years. To provide a full schedule of discount factors, we need a thirty-year yield curve estimated on a daily basis.

Estimation of yield curves is an important area of research in finance and much progress has been made over the years. McCulloch (1973) uses cubic splines, Fong and Vasicek (1982) use exponential splines. However both of these approaches exhibit some shortcomings at the long end of the yield curve so Nelson and Siegel (1987) developed a parsimonious yield curve fitting model to estimate a continuous yield curve. In practice there is not a lot of difference among them, but Nelson and Siegel (1987) does offer some marginal improvement over other methods so we chose to use Nelson and Siegel (1987) to estimate the UK Treasury yield curve on a daily basis from January 1996 to December 1998.

Common to all yield curve estimation techniques, Nelson and Siegel (1987) estimate a discount factor function that provides a best fit to zero coupon yields implied by the observed Treasury coupon bond prices. This discount function provides zero coupon discount factors for any maturity by simply choosing the corresponding value for the time parameter. We obtain this discount function by estimating zero coupon discount factors from coupon bonds through a technique called "bootstrapping".

To see how this is done, consider a six-month T-bill that yields 5%, and a one-year 5.25% semi-annually paid Treasury coupon bond priced at £100. The six-month T-bill yield of 5% is a zero yield since only one payment is made at maturity. The yield on the one-year coupon bond is not a zero yield since £2.625 is paid in six months, and £102.625 is paid at the end of one year. So the question becomes, what is the true zero yield in one year's time when we know the six-month zero yield is 5%? To answer this question, examine the pricing equation of the one-year bond:

$B = C.d(1) + [C+M].d(2)$

where C is the coupon, M is the principal repayable at maturity, and d(1) and d(2) are the existing (non-stochastic) one and two period discount factors implied by the true existing zero coupon yield curve. However, we know all parameters except the two-period discount factor d (2). Since we have one equation and only one unknown, we can solve the above equation for d (2).

$£100 = 2.625*(1/1.025) + [2.625+100]*d\ (2)$

so d(2) is 0.94947.

To check, $2.625*(0.9756) + 102.625*(0.94947) = 100$.

Similarly, we can find the three-period zero coupon discount factor by using the price of a three-period (18-month) Treasury coupon bond as follows:

$B = C*d(1) + C*d(2) + [C+M]*d(3)$.

Again we have a pricing equation with only one unknown - d(3) - since we have previously solved for d(2). It is a simple matter to solve for d(3). We can continue to solve for later maturity discount factors recursively using successively longer-term coupon bond prices.

Essentially Nelson and Siegel (1987) uses this bootstrap method to abstract from coupon bond prices the raw data (observed zero coupon discount factors) to estimate a zero coupon discount *function*. Once estimated, the discount function provides zero coupon prices continuously for all maturities up to 30 years. From these discount prices we then abstract the corresponding zero coupon yield to maturity for all possible maturities. In this way we obtain a continuous zero coupon yield curve. We use three and six-month T-Bill rates along with yields on all actively traded (large issue size) Gilt bonds as the raw data so we avoid the inaccurate data problem as well.

3.2 Volatility Issues

The second empirical issue is an accurate estimate of the volatility of interest rates. This is important because the volatility parameter is used to generate the distribution of all possible interest rates we use to value the redemption feature.

It is common market practice to use an implied volatility measure estimated from "turning around" Black and Scholes (1973)[2]. In other words, rather than estimating volatility to solve for the market price of an option, we accept that the market price is correct, and solve Black and Scholes (1973) for the unknown volatility parameter. This is in contrast to using a historical estimate of the variance of the short interest rate.

Empirical tests have shown that if one were to use a historic measure of volatility, the price thereby obtained would not be as accurate as the current market price. These tests conclude that market participants anticipate future volatility in the pricing of options. Hence, if one were to accept that the market price of an interest rate option is correct then one can solve the Black and Scholes (1973) model for the unknown volatility to obtain a more accurate estimate of volatility than a simple measure of historic volatility. This is the procedure we employ. In practice this is very easy to do since interest rate caps and floors are quoted in term of volatility. This information is readily available on *Datastream*.

[2] Strictly speaking market practitioners "turn around" the model contained in Black (1976), a variation of the model derived in the celebrated Black and Scholes (1973).

4.0 PRACTICAL APPLICATION

There are a number of issues associated with applying the Black Derman and Toy (constant volatility version) to valuing redemption mortgages. First, we must choose an appropriate yield curve that is applicable to the UK retail bank market. In doing so, we must bear in mind that the essence of banking is to earn money from borrowing at low interest rates, and lend money at higher interest rates, thus earning money from the spread. This implies that the retail bank market faces two yield curves, a borrowing and a higher lending yield curve. Furthermore, UK retail banks are subject to credit risk implying they cannot borrow at UK sovereign rates, so even the lower borrowing yield curve must lie above the UK sovereign yield curve. Second, we must choose a measure of volatility that is applicable to the UK bank market. Last but not least we must work out how payments on UK mortgages are charged. Each of these issues is discussed in turn.

4.1 Estimating Bank Borrowing and Lending Yield Curves

Since UK financial institutions are subject to credit risk, it would be inappropriate to calibrate our model to the UK Treasury yield curve. The resulting Arrow-Debreu prices (and discount factors) would be appropriate to value those financial securities that are not subject to credit risk, which is not the case here. Hence, without any adjustment, use of the Treasury yield curve would result in understating borrowing and lending rates and overstating the value of financial assets.

A solution is suggested by the close correlation among the Base Rate, quoted variable mortgage rates and short-term Treasury interest rates. Table 2 shows the correlation matrix among these interest rates for the UK clearing banks' base rate and middle rate, UK Treasury one-, two- and three-month T-bill rates, and the variable mortgage rates quoted by Abbey National, Alliance & Leicester, Halifax, Nationwide and Woolwich. This information is collected from *Datastream* and covers the 1989 to 1998 calendar years on a weekly basis.

Table 2
Short-term Interest Rate Correlation Matrix

	Base	T-B 1 Mo	T-B 2 Mo	T-B 3 Mo.	Abbey	A&L	Hal.	Nat. Wide	Wool.
Base	1	0.8084	0.8661	0.8938	0.9290	0.8880	0.9550	0.8004	0.9214
T-B 1 Mo.	0.8084	1	0.9300	0.8653	0.7102	0.7667	0.7180	0.5121	0.6509
2 Mo.	0.8661	0.9300	1	0.9746	0.7683	0.8389	0.7789	0.5418	0.7033
3 Mo.	0.8938	0.8653	0.9746	1	0.8051	0.8491	0.8213	0.5875	0.7468
Abbey	0.9290	0.7102	0.7683	0.8051	1	0.9216	0.9827	0.8785	0.9453
A&L	0.8880	0.7667	0.8389	0.8491	0.9216	1	0.8873	0.7007	0.8315
Hal.	0.9550	0.7180	0.7789	0.8213	0.9827	0.8873	1	0.8876	0.9563
Nat.	0.8004	0.5121	0.5418	0.5875	0.8785	0.7007	0.8876	1	0.9015
Wool.	0.9214	0.6509	0.7033	0.7468	0.9453	0.8315	0.9563	0.9015	1

Table 2 shows that the correlation between quoted variable mortgage rates, which represent the UK retail banking industry's short-term lending rate, is highly correlated with short-term UK T-bill rates[3]. The base rate, which represents the UK retail banking industry's borrowing rate from the UK Treasury, is also highly correlated with short-term T-bill rates. Since these short rates are so strongly associated, it seems reasonable to suggest that the unobserved UK retail banking borrowing and lending yield curve moves with the underlying Gilt yield curve.

Consequently we chose to calibrate the Black Derman and Toy (1990), constant volatility version, to the UK Gilt yield curve plus a spread. For the lending yield curve, the spread would be the difference between the quoted variable mortgage interest rate (at the time the mortgage is taken out) and the three-month T-bill rates. We chose three-month T-bill rates as Table 2 shows that the correlation between this rate and all quoted variable mortgage rates are consistently higher than alternative T-bill rates. The short-term interest rate would be the one-month T-bill rate plus the spread. From then on, all interest rates we use to calibrate the stochastic interest rate model would be the observed Gilt yield plus the spread between the observed variable mortgage rate and the three-month T-bill yield.

We can apply an analogous method when estimating the retail bank's borrowing yield curve. That is the first interest rate will be the one month T-bill yield plus the spread between the base rate and the three-month T-bill yield. From then on, we use the Gilt yield curve plus the spread between the base rate and the three month T-bill rates for all other interest rates along the bank's borrowing yield curve. However, we believe this would be an upwardly biased estimate of the bank's borrowing yield curve for two reasons. First, only a small portion of a bank's borrowing is from the Treasury. Second, a special characteristic of the retail banking industry is its ability to borrow money through retail deposit taking at extremely low rates of interest. Hence a large and unknown portion of their funding is from non-financial market based borrowing, where safekeeping, record keeping and convenient payment systems compensate the deposit holder in a non-pecuniary way. Nevertheless, we believe an upwardly biased borrowing yield curve will be useful since it can give up a downwardly biased estimate of the bank's gross profit from mortgage lending. As long as we are careful to interpret the figure as *at least* the gross profit earned by the financial institution on a particular mortgage, a useful figure is provided.

Notice that these yield curves imply that the financial institution earns a spread between the implied borrowing and lending rates that is constant with maturity. This may not be the case. However, the only way to improve on this methodology is to require the financial institution to reveal its borrowing costs. As this is a key competitive variable, banks would be reluctant to do this. Furthermore, it is doubtful that the banks can even do this in any precise way since banks do not actively borrow or lend money at many points along the yield curve. It is hard to see how we can improve upon the assumption of a constant earned spread all along the yield curve.

However, we can judge the reasonableness of this assumption by noting that the value of the mortgage amount using the discount factor tree found by the lower borrowing yield curve will be higher than the actual mortgage amount. The difference would represent an estimate of the present value of the minimum gross profit earned by the

[3] This is a short-term rate because borrowing at floating rates in effect creates a zero coupon loan on a periodic basis. In other words, a variable rate loan is a periodic zero coupon bond. The periodicy of the zero is determined by the periodicy of the reset dates, when the variable rate is reset to some new value.

bank on a particular mortgage. We would certainly expect to see a positive gross profit at initiation of the mortgage, and a likely profit of a few thousand pounds per £100,000 if the methodology is reasonable.

4.2 Volatility

As suggested earlier, we intend to use interest rate cap volatility as the measure of volatility that we use to generate the distribution of short interest rates in the Black Derman and Toy (1990) model. Since our short rate of interest is the one-month rate we will use the short (one year) interest rate cap volatility[4]. For August 14, 1996, this annualised interest rate volatility was 15%.

We have found above that all short rates of interest are extremely highly correlated, so we feel confident that the market determined interest rate cap volatility is applicable to the lending and borrowing retail bank yield curves. However to check on the reasonableness of this assumption, we also calculated the interest rate volatility from the historic time series of the bank rate. We did this twice on a weekly basis, once for the ten-year period for the calendar years of 1989 to 1998 and again for the five-year period for the calendar years 1994 to 1998. The later period was chosen since the ten-year sample appears to incorporate two distinct sub-periods where the later half of the period enjoyed much less volatility. This is shown in Appendix 2A.

We found that these alternative measures of volatility (standard deviation) were different from the value obtained by the interest rate cap, ranging from 46.6% for the whole ten-year period to 9.5% for the last five years of the sample period. We then applied these alternative measures of volatility to our numerical example and found that the value of the redemption feature did not change very much. This is not surprising since the interest rate distribution generated by the model is symmetrical. The larger the standard deviation, the larger and smaller are the extreme outcomes. However, most of the value of the derivative is still controlled by the middle interest rate states since extreme outcomes are unlikely. At any interest rate volatility level, these middle interest rate levels are similar. Hence the value of the redemption feature is not extremely sensitive to the value of the interest rate volatility. Since interest rate cap volatility is a theoretically superior measure of volatility, we chose to use this measure of interest rate volatility.

4.3 Payments on UK Mortgages

It is important that we calculate mortgage payments required by UK financial institutions correctly. The incentive to repay a mortgage early and incur the redemption fee is dependent upon the savings captured by lower mortgage payments upon refinancing at lower mortgage rates. Therefore, unless we calculate the old and new mortgage payments correctly, we will not calculate the incentive to refinance correctly either.

[4] This is the shortest interest rate volatility available on Datastream.

UK mortgage payments are calculated in an odd way. First, the bank or building society will calculate the *annual* annuity necessary to repay the mortgage in the number of years the mortgage is to last. This is like assuming consumers will repay, say 25-year mortgages, in 25 annual payments. Of course they actually make monthly payments, so the bank then divides the annual annuity by 12 to obtain the monthly payment. Obviously this raises the interest cost, but on the other hand the effect of this in included in the APR, so consumers are informed of the true cost of the mortgage.

A second oddity is that at the end of the first calendar year, reconciliation is made between the annual interest cost charged and the number of payments made. This adjustment is necessary because the above mortgage payment calculation assumes an annual annuity that is paid in 12 monthly instalments. Therefore, if the mortgage begins at any month other than the first month of the bank's year, less than 12 monthly payments would have been received for the bank year, yet a full year's interest would have been charged. Here the bank will pro-rate the interest charged for the number of days the mortgage has been outstanding. Then they will credit the total balance charged by the number of payments actually made divided by twelve. That is,

Reconciliation = m*i*days/365 – m*i* No. of Payments/12

where m is the borrowed amount, i is the stated interest rate and days are the actual number of days the loan is outstanding during the year. For example, suppose we borrow £100,000 at 8% stated interest on September 12. So we make payments on October 12, November 12 and December 12. Therefore,

Reconciliation = £100,000 * 0.08 * 111/365 -£100,000 * 0.08 * 3/12
= £432.88

Since interest is charged on a daily basis but payments are credited on a monthly basis, this results in an additional charge of a few hundred pounds. This amount is added to the balance at the end of the first year. It means that from the date of reconciliation onwards, the "fixed" mortgage payment increases slightly because the balance owed increases. Again however, this additional cost is reflected in the APR.

5.0 VALUING REDEMPTION FEATURES

We now discuss how we are to apply the stochastic interest rate model to the practical problem of valuing redemption features. First we give a conceptual overview pointing out that the redemption feature gives the mortgage holder an American call option. Then we go through the details of how we are to price this American call option.

5.1 Overview

Whenever a consumer obtains a fixed rate mortgage, they implicitly buy a call option. This call option allows them to call (or buy back) the mortgage at face value (the remaining balance thereof) when the mortgage is actually worth more than face value. The mortgage is worth more than face value because the consumer is making a monthly payment that is higher than would be paid using current lower mortgage rates. On the other hand, if mortgage rates increase, they will not exercise their call option because they will in effect be buying back a less valuable mortgage at the higher face value.

From this point of view we can see that indeed the consumer has a call option with a pay-off function of Max {S-X; 0). S is the underlying asset or the value of the mortgage and X is the exercise price, which is equivalent to the remaining balance owed on the mortgage. For example, if they borrow £100,000 for 25 years at a fixed rate (for five years) of 8% on January 1 of the year, and mortgage rates were to immediately decrease to 7%, they can profitably buy back the mortgage at £100,000 and refinance at the lower 7% rate. Notice that the option matures in five years because at that point the mortgage would be renewed at the current fixed or variable rate, whatever the consumer chooses.

It is most convenient to work out the pay-off on this call option as the difference between the value of the monthly payments for five years at 8% and five years at 7%. Then we value the monthly saving as a five-year annuity discounted at the current five-year mortgage rate. That is,

$$\frac{m}{\left[\dfrac{1-(1+i_h)^{-n}}{i_h}\right]} \div 12 - \frac{m}{\left[\dfrac{1-(1+i_l)^{-n}}{i_l}\right]} \div 12$$

Where i_h is the higher mortgage rate, i_l is the lower mortgage rate, and m is the mortgage amount. Then we value the annuity as,

$$Pmt\left[\frac{1-(1+i_l/12)^{-np}}{(i_l/12)}\right]$$

where Pmt is the mortgage cost saving and np refers to the number of payments remaining till the loan is renewed.

In this case, the monthly amount saved by re-mortgaging at 7% is

$$\frac{100{,}000}{\left[\frac{1-(1+.08)^{-25}}{.08}\right]}\div 12-\frac{100{,}000}{\left[\frac{1-(1+.07)^{-25}}{.07}\right]}\div 12$$

= 780.65 – 715.09

= £65.56

The value of the mortgage payment saving is,

$$65.56\left[\frac{1-(1+0.00583333)^{-60}}{(0.00583333)}\right]$$

= £3,310.91

This is the *intrinsic* value of the put option at the present time when mortgage rates immediately decrease to 7%. The intrinsic value refers to the value of an option if it was to be exercised immediately and would represent the pay-off at maturity of a European option. However, mortgage holders have an American option since they can choose when to refinance - they are not restricted to calling the mortgage at a particular date.

Mortgage holders may rationally decide to delay exercising their call option because they anticipate that next period, interest rates may decrease even further. Since pay-offs on interest rate options are a non-linear function of the interest rate level, they may obtain a larger risk adjusted gain if they delay call till some future time when interest rate levels are lower.

For example, suppose a mortgage holder has borrowed £80,000 at 7.85%. Now interest rates are 7.77%, meaning that if 60 months remain on the fixed rate period, they can save £4.24 per month or save £210.25 in present value terms. However, next period, interest rates may decrease to 7.52%, or increase to 8.19% with equal probability. If interest rates rise their option is no longer in the money (it has an intrinsic value of zero). If interest rates decrease the required monthly payment decreases to £599.12, a monthly saving of £17.42. On a present value basis, this 59-period annuity is worth £856.96. If we rolled back through the tree one month to today, we find that the value of the choice to delay call by one month is 0.5(0 + 856.96)/(1+0.0777/12) = £425.72, more than twice the current intrinsic value of the option. Therefore the mortgage holder may rationally delay call even though the option currently has a positive intrinsic value.

5.2 Valuing the American Call Option

Therefore our challenge is to value an American option. This is done in a two-step process. First, we work out the intrinsic value of the option to repay early at every branch of the interest rate tree. This intrinsic value is calculated as the present value of the annuity of the mortgage savings captured by refinancing at a lower rate as demonstrated above. At any time the interest rate tree generates a mortgage rate that is higher than the current mortgage rate, the intrinsic value is zero. Hence we develop an intrinsic value tree similar to Figure 5, only the values at each node represent the intrinsic value of the option to refinance at that node.

Second, we Americanise the intrinsic values by considering at each node whether the option is worth more alive by waiting one more period and hoping to obtain an even larger intrinsic value next period, or dead by exercising immediately and capturing the current intrinsic value. This is done by solving backwards. That is, we begin at one month prior to the end of the fixed rate period. At this point in time, the option to refinance is European, since the option is now at its maturity. Automatically, the value of the option at this point is its intrinsic value. Then we consider the value of the option two periods prior to the end of the fixed rate period. We calculate its value if we were to delay one more period (its American value) and compare it to the previously calculated intrinsic value (its European value assuming this is the expiry date). Its American value would be one half of the sum of its intrinsic value in the adjacent next period (at maturity of the option to call) interest rate states discounted back by the same node discounted factor generated by our calibrated interest rate tree. We take the higher of these two values. This is exactly the same procedure as used in the example in Section 5.1. We do this for every node of the American option tree two periods prior to the end of the fixed rate period, where every node corresponds to an interest rate level as generated by our calibrated interest rate tree.

We continue to follow this two-step process, rolling backwards through the tree. For example, after we have calculated the American value of the option at two periods prior to the end of the fixed rate period, we then consider the value of the option three periods prior to the end of the fixed rate period. We again take the higher of its intrinsic values as previously calculated or its American value. Again, the American value is one half of next period's (now two periods prior to the end of the fixed rate period) adjacent states option values as found in the prior iteration, discounted back to the beginning of the period. Again our calibrated interest rate model generates the discount factor. We continue solving backwards in this manner for as many months as there remains in the fixed rate period until we find today's value of the American option implied by the mortgage holder's ability to redeem the mortgage early.

Once we have calculated the value of the option to refinance we compare it to the redemption charge. We would expect that the value of the redemption charge will equal the value of the option implicitly given to the mortgage holder when agreeing to a fixed rate mortgage, plus a modest administrative cost premium of perhaps £200, £300 or £400.

As discussed in Section 1.1, we ignore the impact of non-financial early repayments. Non-financial early repayments may represent some cost recovery for the bank since some early repayments may occur at disadvantageous (from the mortgage holder's perspective) mortgage rates. In the absence of any policy of waiving redemption fees in this case, then we would expect that the redemption fee (including an administration fee) would be only slightly higher than the value of the call, which ignores non-financial redemption. This means that if the redemption fee were much higher than our calculated call value then we would have found evidence that the financial institution is overcharging for redemption.

6.0 APPLYING THE MODEL-NUMERICAL EXAMPLE

We are now ready to apply the arbitrage free stochastic interest rate model to value the redemption feature. We keep in mind the solutions to the empirical and practical problems discussed in Sections 3 and 4.

We value the redemption feature of a 25-year fixed rate repayment mortgage taken out on 14 August 1996. The relevant details are as follows. The mortgage amount is £82,474.06, and the fixed rate is 7.85% (APR 8.3%). The mortgage rate is fixed until 30 April 2001 implying fixed mortgage payments for 57 months. The initial non-MIRAS mortgage payment is £635.61. That is

$$\frac{82{,}474.06}{\left[\dfrac{1-(1+.0785)^{-25}}{.0785}\right]} \div 12 = £635.61$$

At the end of the bank's financial year, the building society will reconcile their books as of 31 March 1997 (their year-end). They take into account that only seven monthly payments have been made yet the above annuity has been calculated, on the basis of a full year's interest being charged for the first year. There is 231 days' interest to be charged from 14 August to 31 March. Therefore as discussed in Section 4, the reconciliation is

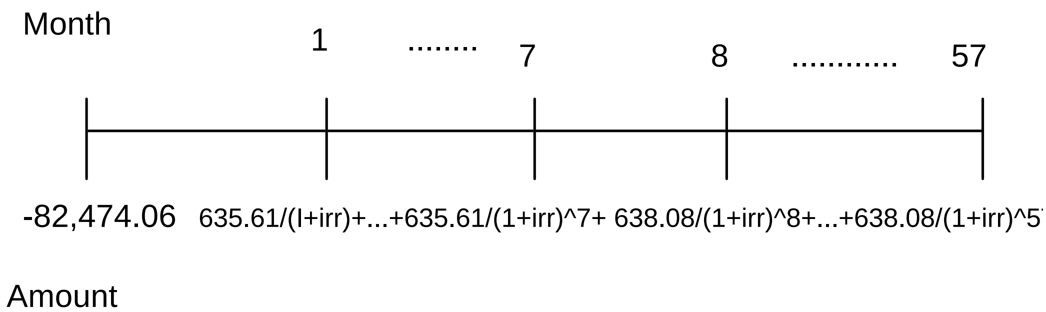
Reconciliation = £82,474.06 x .0785 x 231/365 -£82,474.06 x .0785 x 7/12
= £320.75.

This amount is added to the original principal to calculate a new mortgage payment that is paid from 14 April 1997. This amount is

$$\frac{82{,}474.06+320.75}{\left[\dfrac{1-(1+.0785)^{-25}}{.0785}\right]} \div 12 = £638.08$$

This is the amount that will be paid monthly till the end of the fixed rate period. We can find the true interest cost of this mortgage as the internal rate of return on this cash flow structure. This is done by iteratively solving for that interest rate that present values these cash flows to equal the mortgage amount of £82,474.06 as demonstrated in Figure 6.

Figure 6-True Interest Cost (IRR)



We find that the true interest cost (IRR) is 8.33%, which agrees with the quoted APR of 8.3% since the APR is the true interest rate truncated to one decimal point.

The redemption charge is of the “step down” type common on callable bonds traded on the bond markets. In particular, five months of interest is charged for early redemption during the first year of the fixed rate period, four months’ interest is chargeable in the second year of the fixed rate period, thereafter three months of interest is chargeable until the end of the fixed rate period. Once the fixed rate period expires, the mortgage reverts to a variable rate mortgage with no redemption penalty until the mortgage is paid off on 14 April 2021.

In cash figures, the redemption fee schedule is shown in Table 3.

Table 3
Redemption Fee Schedule

Date	Month	Monthly Interest	No. of Months Interest	Redemption Fee
14-Sep-96	1	539.518	5	2697.59
14-Apr-97	8	537.216	5	2686.08
14-May-97	9	537.216	4	2148.86
14-Apr-98	20	529.298	4	2117.19
14-May-98	21	529.298	3	1587.89
14-Apr-99	32	520.758	3	1562.28
14-Apr-00	44	511.548	3	1534.65

Notice that there is a minor “step down” in the redemption charge in April of each year since the building society credits principal paid during the year at the end of March. Hence, since the principal is smaller, the monthly interest chargeable is smaller and so too is the repayment charge.

The implied call option expires one month prior to the end of the fixed rate period, so the maturity of the implied option created by the ability of the mortgage holder to redeem prior to the end of the agreed term is 56 months. What is the value of this American option to the mortgage holder?

First, we must realise that the mortgage holder faces the bank's lending curve. It is only at the bank's lending curve that the mortgage holder can refinance. Therefore we must estimate the bank's lending curve for 14 April 1996. On this date, the variable mortgage rate offered by this building society was 6.99%. Since the three-month T-bill rate was 5.5%, this implies a spread of 1.49%.

We applied Nelson and Siegel (1987) to all straight (no callable, variable interest, inflation protected and so on) Treasury bills and Gilts with an issue size of £50 million or more outstanding on 14 August 1996. As explained in Section 3, this model estimates the implied zero coupon Treasury yield curve that existed on 14 August 1996. We then constructed the lending curve as follows. The one-month T-bill rate was 5.4%, and adding the 1.49% spread, the one-month lending rate is 6.89%. Similarly, the two-month T-bill rate was estimated to be 5.45% so with a 1.49% spread, the two-month lending rate is 6.94%. In this way we continue to add the spread to the underlying Gilt yield curve to construct the existing retail bank lending yield curve.

The annual interest rate volatility that was applicable to one-year interest rate caps was 15%. As discussed in Section 4, this is our estimate of the bank's lending interest rate volatility on 14 August 1996. We use 6.89% as the current short interest rate, along with a 15% volatility estimate, to generate a log-normally distributed stochastic interest rate tree. This tree is used to construct discount factor and Arrow-Debreu trees. We then calibrate the interest rate process, which in turn implies a calibration of the discount factor and Arrow-Debreu trees, by adjusting the calibration factor u_t. We continue to change (calibrate) the factor u_t in our interest rate process until addition of the same period Arrow-Debreu prices replicates the price and yield of the zero coupon bonds that underlie our estimate of the existing retail bank lending yield curve.

Once the calibration process is complete, we now price the implied call option given to the fixed rate mortgage holder as of 14 August 1996. This is done according to the procedure explained in Section 5. First, we work out the intrinsic value of the option to redeem the mortgage prior to the end of the agreed term as the value of the cost-saving annuity implied by refinancing at each interest rate state. If future mortgage rates were lower than the fixed rate (7.85%), the intrinsic value would be positive. If future mortgage rates were larger than fixed rate, then the intrinsic value is zero (See Section 5.1). We then "Americanise" the intrinsic value by "rolling backwards" through the call option price tree according to the procedure as explained in Section 5.2. In other words, we examine the value of the option at each time and state (node of the tree) to see whether the intrinsic value obtained by immediate exercise at that node, or delaying call until the next period is higher. We always take the higher of the two. We eventually work our way backwards through the tree to arrive at the present time. We find that the value of this redemption feature was worth £2,595.45 (See cell C411 in the electronic spreadsheet - for details of how to obtain a copy of the spreadsheet see page 2).

Comparing the value of the implied option to the redemption fee, we find that the redemption fee was £2,697.59. This is £102.14 higher than the call value given to the mortgage holder. The small overcharge probably reflects an administrative charge.

6.1 Implicit Redemption Fees

So far we have examined the *explicit* redemption fee charged by the financial institution in the event of an early redemption by the mortgage holder. However, the financial institution has another adjustable parameter that it can set to extract an *implicit* redemption fee, namely the fixed mortgage rate. In this case, the bank may forecast that interest rates are to change. Say they expect interest rates to rise. Then they would quote a higher fixed interest rate, but perhaps a modest redemption fee. Implicitly the redemption fee is higher than the explicit fee since at least part of the incentive to refinance at a later date (should interest rates rise then fall) has been charged by paying fixed rates of interest at rates that were too high initially.

The idea here is that the bank can charge a fixed mortgage rate that is above the average expected lending curve. Since the fixed mortgage rate is larger than the interest rates implied by the average (calibrated) discount rates derived from the existing lending curve, we would find that the value of the fixed rate mortgage is above the actual mortgage amount advanced by the financial institution. The difference between the mortgage value found by the calibrated interest rate model and the amount borrowed would be the value of the implicit redemption fee[5].

In this example, we discover that the calibrated interest rate model values the mortgage as £80,866.88 yet advances £82,474.06, implying a £1,607.18 loss to the bank (see cell B288 in the spreadsheet - details of how to obtain a copy of it are on page 2). Rather than charge an implicit redemption fee, the financial institution offered an implicit mortgage incentive. This implicit mortgage incentive is a measure of the competitiveness of the mortgage offer as of 14 August 1996, for it represents a discount from the normal lending curve yield the bank was willing to give in order to sell the mortgage.

If we were to value the mortgage using our estimate of the borrowing yield curve we would arrive at a present value estimate of the gross profit expected by the bank on 14 August 1996. The spread between the bank rate and three-month T-bills was 25 basis points on that date. Using this spread, rather than the 149 basis points used to calculate the lending yield curve, we find that the bank will value the mortgage as £84,724.73, implying an estimated minimum gross profit of £2,250.67.

[5] This is a much more intuitive way of explaining how to measure the implicit redemption fee that was used in the proposal. However the swap analogy still applies. The floater is the corresponding variable rate mortgage that will be valued at the mortgage amount since the floating rates will change to reset the mortgage to par (remaining amount borrowed) at each reset date. In the example, the floater F = £82,474.06. The fixed rate bond is the fixed rate mortgage. In the example, the fixed rate bond B is £80,866.88. Since the bank is in effect long a fixed rate mortgage and short a variable rate mortgage, the value of the swap S from the bank's perspective is S = B-F or –1,607.18. This is the same as in the text.

7.0 MORTGAGE INCENTIVES

Most UK building societies and banks offer mortgage products with some kind of mortgage incentive. While the exact nature of the incentive varies widely from product to product, conceptually they seem to be of only two stylised types, the “cash back” and the “discounted mortgage”. With the cash back mortgage, the homeowner takes out a mortgage in the usual way, but shortly after the mortgage has been arranged, the bank pays a specified amount to the mortgage holder. In return, the mortgage holder must continue to service the mortgage until some pre-specified date. Early redemption within this time period triggers a redemption penalty equivalent to the amount received as a cash back. With the discount mortgage, the homeowner arranges the mortgage in the usual way, but the financial institution will charge a lower interest rate for a specified time period. Like the cash back, early redemption within a pre-specified time period triggers a redemption fee supposedly equivalent to the value of the discount.

We will examine an example of each of these mortgages below. Unlike the redemption fee example examined above, these examples are based on variable rate mortgages. The variable rate mortgage presents no particular difficulties for the Black, Derman and Toy (1990) model as it is specifically designed to handle stochastic (variable) interest rates. However, calculations of mortgage values are complicated because the final amount owing once the incentive period ends is *path dependent.* This happens because as the variable rate changes, not only does the interest charge change, but also that portion of each payment that represents a repayment of principal. To see why this is the case, consider the following illustration.

Suppose you borrow £157,000 at 8.05% variable for 25 years. The first payment would be,

$$\frac{157{,}000}{\left[\dfrac{1-(1+.0805)^{-25}}{0.0805}\right]} \div 12 = £1{,}230.87$$

This amount is composed of £1,053.21 interest and £177.66 principal repayment. That is, £157,000 (1+.0805/12)-£157,000 = £1,053.21 so payment less interest is the principal repayment. The mortgage balance would then be £157,000 (1+.0805/12)-1,230.87 = £156,822.34.

Now suppose that next period interest rates were to increase to 9.05%. Then the new repayment would be,

$$\frac{156{,}822.34}{\left[\dfrac{1-(1+.0905)^{-25}}{0.0905}\right]} \div 12 = £1{,}335.85$$

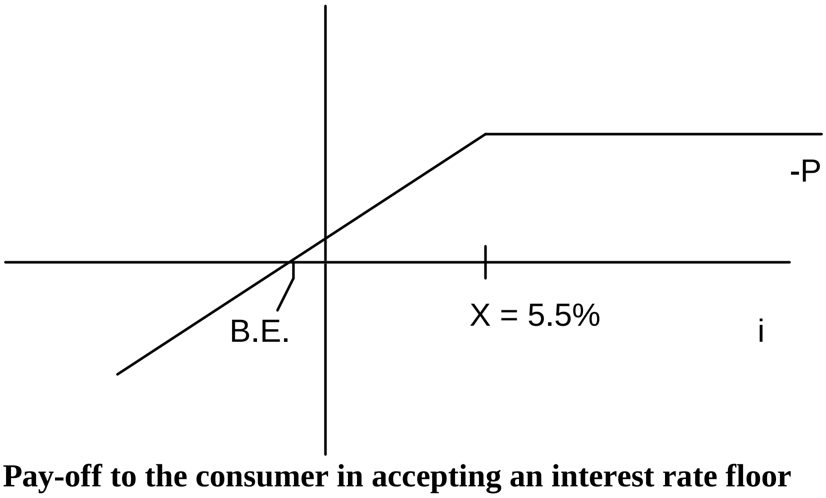
The mortgage amount remaining would then be £156,822.34 (1+0.0905/12)-1,335.85 = £156,669.19. In contrast, if interest rates decreased to 7.05% in the second month, the amount owing at the end of the second payment would be £156,617.20.

Now consider the recombining state at the end of the third period. If we move from the higher interest rate state of 9.05% back to 8.05%, the amount owing at the end of the third period would be £156,491.90. However, if we move from the low interest state of 7.05% back to 8.05%, the amount owing would be £156,439.97. We see that in interest rate state S (2,1), the amount owing could be £51.93 higher if we have moved from the higher to the lower interest rate level rather than from the lower to higher interest rate level. This happens even though we end up at the same interest rate level (8.05%). So we see that path dependence means that the amount owed after a period of time will depend upon the sequence of past variable mortgage rates charged. This is important in the examples below because we must find the amount owing on the mortgage at the end of the early redemption period in order to determine if the mortgage and the redemption penalty is valued correctly.

A final feature of this section is that we value an interest rate floor. An interest rate floor is a put option held by the financial institution. The idea is as follows. The homeowner agrees to “floor” the variable mortgage rate at a minimum level. The interest rate charged could vary as the variable rate changes, but when the variable rate reaches the “floor”, say 5.5%, then the mortgage holder agrees to pay 5.5%, even if the unrestricted variable rate continues to fall below 5.5%. Of course, if interest rates subsequently rise above 5.5%, the mortgage holder will pay the higher rate.

From the homeowner's perspective, they have sold an interest rate put option to the financial institution. To see this, the following pay-off diagram is helpful (Figure 6).

Figure 6



Pay-off to the consumer in accepting an interest rate floor

The x-axis notes the current variable mortgage rate where the exercise price is 5.5%. The y-axis plots the amount the mortgage holder gains/loses at expiry of the option. Since they are selling a valuable security to the financial institution, the mortgage holder should receive payment for agreeing to sell a floor. The distance between the x-axis and the put pay-off diagram (-P) at i = X represents this payment. If at expiry, the variable rate of interest is above the floor, the option expires worthless and the mortgage holder keeps the full amount of the premium. On the other hand, if the variable rate of interest is below the exercise price, then the mortgage holder must continue to pay the exercise price interest rate of 5.5%, thus losing value. Below B.E. in Figure 6 the spread between the higher exercise interest rate and the unrestricted variable mortgage rate is so large that the mortgage holder experiences losses greater than the initial premium received for selling the option.

From the above discussion it is obvious that a *capped* mortgage is an ordinary variable rate mortgage with an interest rate call option held at the hands of the mortgage holder. The idea here is that as interest rate rises, the mortgage holder would continue to pay the higher interest rate until the exercise interest rate is reached. From then on, the cap is binding and the mortgage holder continues to pay the (lower) exercise interest rate even if the unrestricted variable mortgage rate continues to rise. Here the mortgage holder receives a valuable security, so they must pay a premium, typically through paying a higher variable mortgage rate at the beginning of the mortgage agreement. We find the value of this interest rate call option in exactly the same manner as we will find the value of an interest rate floor, only the signs of the cash flows will be opposite.

In the next subsection we discuss how to model a variable rate mortgage. We then examine two examples. The first is a cash back variable rate mortgage with an interest rate floor. The second is a variable rate mortgage with a discounted interest rate.

7.1 Variable Rate Mortgages

As always the first few steps are to estimate the existing sovereign and lending yield curves and then calibrate an interest rate process such that the structure of Arrow-Debreu security prices obtain yields that agree with the existing lending yield curve. We use this information to price the following variable rate mortgage.

This mortgage was agreed on 31 December 1998. This is a 25-year, variable rate repayment mortgage for £157,000. The initial mortgage rate was 8.05%, and was floored at 6.75%. The mortgage holder received 6% of the mortgage amount plus £250 as a cash back incentive, or £9,670. If at any time the mortgage holder repays the mortgage before 31 January 2004, then the mortgage holder must repay the cash back. Repayments for partial early redemption are calculated on a pro-rata basis. Therefore our challenge is to value a 61-month variable rate repayment mortgage.

Jumping ahead slightly, we consider the methodology of how to value the entire package of securities that comprise the above example. We have argued that the floor represents a short position (from the mortgage holder’s perspective) in an interest rate put option and they are obviously short in a straight (no options attached) variable rate mortgage. Hence we can view the value of the package as a combination of two securities as follows:

Mortgage Advance + Cash Back = Straight Variable Rate Mortgage + Put Option.

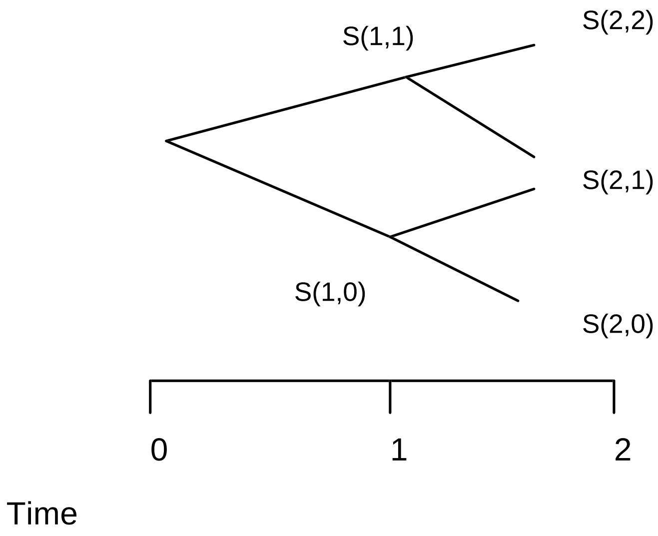
Note that the left-hand side of the above equation represents the value of the mortgage from the mortgage holder's perspective and the right-hand side represents the value of the same offer, but from the financial institution's perspective. Hence our methodology is first to value the unrestricted variable rate mortgage (straight mortgage) and then value the put option. Together these two securities should equal the mortgage advance plus the cash back payment. Note however, it is not necessary that the put option value equals the cash back, or that the mortgage advance (£157,000) equals the straight mortgage value. For example, the financial institution can implicitly claw back part of the value of the cash back through under-advancing the true value of the underlying straight variable rate mortgage[6]. Next we value the straight mortgage. We defer valuing the put option until the next subsection.

The first step is to find the amount owing as of 31 January 2004. As explained above, the amount owing on 31 January 2004 will depend upon the sequence of variable rates experienced during the prior 61 months. With a monthly time step, our binomial interest rate process will generate 61 possible values of the mortgage as of 31 January 2004, 61 months from 31 December 1998. These values will cover all possible values of the mortgage because we use a *recombining process* that forms a *Markov chain*. As we have mentioned before, a recombining interest rate process means that an up interest rate movement followed by a down interest rate movement leads to the same interest rate state as a down-then-up sequence in interest rates. However, the mortgage values will be path dependent, so the value obtained from an up-then-down sequence in mortgage rates will in general *not* be the same as the value obtained through a down-then-up sequence in interest rates. How do we account for path dependence?

Our process is a Markov chain, so the value obtained next period depends upon the two adjacent current interest rate states, and the interest rate state that evolves next period. To see this consider the following diagram (Figure 7).

[6] That is, advance, say, £157,000 to the mortgage holder of a mortgage that is worth £160,000 thus clawing back £3,000 of the cash back.

Figure 7-Path Dependence



The amount owing in states S (1,1), S (1,0), S (2,2) and S (2,0) are unique since there is only one sequence of variable interest rates that can occur to obtain the value of the mortgage amount in these states. However, for interest rate S (2,1), the mortgage amount obtained would depend on whether interest rates have first increased (S (1,1)) and then decreased (S (2,1)) or first decreased (S (1,0)) and then increased S (2,1)). In general, the mortgage amounts thereby obtained will be different.

Notice that the amount owing obtained in state S (2,1) does not directly depend on the starting interest rate at S (0,0) since the impact of this would be summarised in the mortgage values obtained in S (1,1) and S (1,0). In other words, we do not need to know the entire history of interest rate movements. The impact of all possible historical interest rate paths are summarised in the prior period adjacent interest rate states. This is the Markov property. Therefore all we need to worry about is the beginning of period mortgage values in states S (1,1) and S (1,0) (which of course depends upon the interest rates that did evolve in these states) and the interest rate state that evolves in S (2,1).

So moving forward in time, we only have to worry about the two adjacent current interest rate states and the immediately following interest rate state. We use the equivalent Martingale measure (pseudoprobability) of 50% (the 50/50 rule mentioned earlier) to account for the probability of a higher or lower beginning of period amount owing, so next period's mortgage value is an expected value[7]. This expected value

[7] How can we assume that the probability of an up or down interest rate movement is equally likely? Because we calibrate our interest rate tree to the current lending curve! Remember that we are calculating "expected values" of the mortgage next period. We can obtain true expected values by adjusting the probabilities to the true probabilities (that is, the Governor of the Bank of England will increase the 5.5% interest rate to 6% next period with 80% probability or reduce it to 5% with 20% probability) or by adjusting the level of possible interest rates next period (next period's interest rates are 6% or 5.6% with equal probability). Notice that our method is to adjust interest rate levels.

then accounts for the up/down and down/up sequences in interest rates. Therefore the 61 terminal expected values of the final amount owing on the mortgage at the end of the redemption fee period represents all 230,584,300,921,000,000 (2^{61}) possible interest rate paths and the corresponding mortgage values that the mortgage holder needs to consider.

Procedurally, we start our principal value tree at £157,000. Next period, the amount owing on the mortgage will be £156,843.70 since the first variable rate is known at the beginning of the period (8.920488%). The amount owing is calculated as

157,000-(157,000/((1-(1+8.920488/100)^-25)/((8.920488/100)/12))-(157,000*(1+8.920488 /100*1/12)-157,000)) = £156,843.70.

To appreciate what the above formula is doing, we will break it down into its parts. The first term represents the beginning of period amount owing. The second term calculates the mortgage payment given that the variable rate is 8.920488% and the mortgage term is 25 years. Note that the mortgage payment includes blended interest and principal repayments. The third term calculates the interest earned during the first month by the financial institution. When the third term is subtracted from the second, the result is that portion of the first payment that is allocated to repayment of principal. This result is then subtracted from the beginning of period amount owing to yield the amount owing at the end of the period.

Note that the above calculation can be computed more efficiently. This calculation is done this way because we can clearly see all the component elements in a logical way. For programming purposes, it is easier to spot errors when the component calculations are clearly laid out, and this is the formula we use in the spreadsheet. In other words, the above formula represents bad maths, but good programming.

Next period, the amount owing may be £156,666.36 or £156,694.49, depending on whether interest rates fall (to 8.055392 %) or rise (to 9.225939%). These interest rate movements occur with equal probability, but of course this interest rate structure agrees with the current lending yield curve. The values are obtained in the down and up sequence of interest rates respectively as

=156843.70-(156843.70/((1-(1+8.055392 /100)^-25)/((8.055392 /100)/12))-(156843.70*(1+8.055392 /100*1/12)- 156843.70)) =£156,666.36

=156843.70-(156843.70/((1-(1+9.225939 /100)^-25)/((9.225939 /100)/12))-(156843.70*(1+9.225939 /100*1/12)- 156843.70))= £156,694.49.

Notice that the only difference between the two states is the variable mortgage rate.

So far we did not need to worry about path dependence since there is only one way to reach each interest rate state. However, the situation changes next period in state S (2,1), where we can reach this interest rate state (and the corresponding amounts owing) two ways, through a down then up sequence of interest rate or through an up then down sequence of interest rates. As you may guess, we calculate this value using an average (using the 50/50 rule) of the prior adjacent state mortgage values in S (1,1) and S (1,0). That is, 0.5*[156,666.36-(156,666.36 /((1-(1+8.351044 /100)^-25)/((8.351044 /100)/12))-(156,666.36 *(1+8.351044 /100*1/12)- 156,666.36))+ **156694.49 -(156694.49 /((1-(1+8.351044 /100)^-25)/((8.351044 100)/12))-(156694.49 *(1+8.351044 /100*1/12)-156694.49))] = £156510.77**

To ease understanding of the above formula, we have highlighted the second mortgage calculation that makes up the average in bold. Noting this, we see that we add together the end of period amount owing if the beginning mortgage value was £156,666.36 and the end of period amount owing if the beginning amount owing was £156,694.49 and then taking an equally weighted average. This is the way we calculate the mortgage amount owing for all the “middle” states as we move forward in time. For all “extreme” states, that is the very highest and lowest interest states at every time period, we simply use last period’s corresponding extreme amount owing as the beginning of period amount. The end result of this process is a column of 61 possible expected amounts owing on 31 January 2004.

Now we can price the variable rate mortgage in exactly the same way we calculate the value of a fixed rate mortgage or coupon bond (See Section 2.26). We solve backwards by taking an average of the adjacent terminal amounts owing plus the current mortgage payment discounted back by the corresponding state discount factor. This finds the value of the mortgage one period prior to maturity (31 December 2003). We continue to solve backwards by taking the average of adjacent mortgage values one period prior to maturity plus the current mortgage payment discounted back by the corresponding state discount factor just as we did when valuing a fixed rate mortgage. This gives us the value of the mortgage two periods prior to maturity (on 30 November 2003). We continue to solve backward recursively until we find that the current value of the mortgage from the bank’s perspective as of 31 December 1998.

However, there is one important difference in the above calculation. Since we are valuing a variable rate mortgage, the monthly payment does vary from state to state as the variable rate changes. Therefore, for each time and interest rate state, the mortgage payment used in the above calculation varies according to the following formula:

$$\frac{157{,}000}{\left[\dfrac{1-(1+r_{i,t})^{-T}}{r_{i,t}}\right]} \div 12$$

Note that $r_{i,t}$ is the variable mortgage rate that evolves in S (i,t), T refers to the number of years remaining (whole years plus part year), and £157,000 is the amount borrowed. The amount borrowed remains the same for all time periods for two reasons. First, because the financial institution requires repayment of the original amount borrowed over 25 years (that is, we are valuing a repayment mortgage). Second, since the year-end of this financial institution is 31 December of the year, and the mortgage happens to commence on this date, we do not have to make an interest rate reconciliation during the first year. In contrast, you will recall we had to do this for the fixed rate repayment mortgage because the mortgage commenced on 14

August but the financial institution’s year-end was 31 March. Finally, note that the mortgage payment will vary as the variable mortgage rate $r_{i,t}$ changes. Otherwise the above formula is exactly the same as the mortgage payment formula introduced in Section 6.0.

When we carry out the above backwards solving procedure, we find that the value of the variable rate mortgage is £157,839.69 from the bank’s perspective. In other words, according to our model the bank is charging an implicit fee of £839.69 for a mortgage security of £157,000 when it is worth £839.69 more.

It is hard to see why this is the case, for a variable rate mortgage does not convey a call option to the mortgage holder. Recall that a variable rate mortgage always requires payment of the current rate of interest so the mortgage holder can never obtain cost savings by refinancing at a lower interest rate. It does not seem likely that the model is in error by such a large amount. However, we note that the volatility of interest rates as of 31 December 1998 seems high by historical standards. To illustrate this, we graphed one-year interest rate caps from 31 January 1995 (the first date that this information is available is on *Datastream*). This information is presented in Appendix A2 in Figure A2. Note that the peak in interest rate volatility of 23.5% is on 31 December 1998, the date this mortgage was agreed.

To check on how sensitive the results are to the volatility parameter, we replicated our results using the same information as before, only with the 31 January 1999 volatility of 16.5%. We find that the mortgage is still valued at £157,690.79, a difference of £148.90. Since a high volatility indicates that the lending yield curve may change by large amounts, we tested to see how sensitive our results are to a shift in the lending yield curve. We again replicated our results using the original information, only now we use the 2 January 1999 lending yield curve. We find that the mortgage is still valued at £157, 861.73, a difference of only £22.04. In any event, our results indicate that the financial institution is charging an implicit administrative fee of more than £200, £300 or £400. In effect, the financial institution has included a “clawback” provision in the mortgage incentive offer.

7.2 Interest Rate Floors

As outlined above, our 31 December 1999 variable rate mortgage example includes an interest rate floor where the mortgage holder must continue to pay a mortgage rate of 6.75% if the unrestricted variable rate falls below that level. As explained above, we can value this feature as a put option.

Cheerfully, it is simple to value this option since it is in essence a sequence of European put options. We will call each European option a “floorlet” and the value of the entire sequence of floorlets represents the value of the floor. For each month a floorlet expires since if the variable rate is above 6.75%, the obligation to pay 6.75% if the unrestricted variable rate is below this level for that month expires forever. Since each floorlet pays-off only at the end of a particular month, they are European options. Hence we can value the entire floor as the sum of the value of component floorlets, one for each month of the mortgage incentive period, 61 in all.

We value the floor using the familiar backward solving procedure. On 31 January 2004 we test to see if our interest rate tree generates an interest rate level that is below 6.75%. If it does not, then the 31 January 2004 floorlet is worth nothing. If our tree generates an interest rate level below 6.75%, we take the difference, divide by 12 (to convert the annual interest rate to a monthly rate) and multiply it by the beginning of period principal that we have measured in Section 7.1. This measures the value of the last floorlet at maturity. We then find the beginning of period value by multiplying this result by the corresponding interest rate discount factor. We do this for all 61 interest rate states that are possible on 31 January 2004. This gives us all possible values of the January floorlet as of 31 December 2003.

We then find the terminal value of the second floorlet maturing on 31 December 2003 in exactly the same manner as above. We add to this floorlet the value of the 31 January 2004 floorlet that we have valued as of 31 December 2003 above. Note that on 31 December 2003, the variable rate may increase or decrease, so we actually add one half of the sum of two possible values of the last floorlet that expires on 31 January 2004 to the terminal value of the 31 December 2003 floorlet. Then we find the value of both floorlets as of 30 November 2003 by discounting the sum of these values back one period using the corresponding discount factor from our interest rate tree. This is exactly the same procedure we used in Section 2.26 to find the value of a coupon bond, only now the "coupon" is the current floorlet, and the two possible "principal values" are next period's adjacent state floorlet values. We do this for all 60 possible values of the two last floorlets that are alive on 31 December 2003. We continue solving backwards in this manner until we find that the value of the sum of these floorlets is £5,474.98 on 31 December 1998.

Why is the floor so valuable? Certainly one important factor is the high interest rate volatility. For example, if the volatility was more modest, say 16.5%, that was observed only one month later on 31 January 1999, the value of the floor would be £1,570.02 lower at £3,904.96. In contrast, the value of the mortgage would be only £148.90 lower at £157,690.79. This highlights the fact that derivatives are much more sensitive to volatility than the corresponding underlying asset and serves as a warning that to estimate reasonable option values one should take care to use an accurate volatility estimate.

Since practitioners observe that interest rate volatility decreases with maturity, one may suggest that the five-year cap volatility should be used. The rationale is that we are valuing an approximately five-year mortgage and floor. However, this would be theoretically incorrect since the mortgage and floor varies with short-term rates. Furthermore we have modelled stochastic interest rates using short-term rates because we have shown (see Section 4.1, Table 2) that the standard variable rate underlying the interest rate floor is highly correlated with the three-month Treasury interest rate. Hence one could argue that we are being generous in using one-year volatility since, strictly speaking, generally higher three months' volatility should be used[8].

Consequently, our best estimate of the incentive value of the cash back was only £3,355.33, a far cry from the stated value of £9,670. Our estimates show that the stated value of the cash back is actually composed of a £5,474.98 premium for selling

[8] The shortest interest rate cap volatility available on *Datastream* is the one-year volatility that we use.

a valuable put option, a £839.69 claw back/administrative fee and only £3,355.33 is a true mortgage incentive. One can argue that a mortgage holder could not reasonably expect that the true incentive could be so high given that they have sold a floor. Yet conversely one can argue that financially unsophisticated mortgage holders cannot reasonably be expected to realise that so much of the cash back represents a premium on a put option being sold to the bank.

Cause for concern?

One may be tempted to suggest that such contracts should be prohibited because mortgage holders are financially unsophisticated and are too much at risk. However, this does restrict consumer choice. A large, fairly-priced floor premium paid up front may be valued by some mortgage holders as a means of financing home improvements, covering moving costs and so on, costs that are evident in the transition from one home to another. They probably do realise that the benefit of obtaining this premium comes at the cost of never having their mortgage rate drop below the floor. To some, this might be a good deal. However, given that mortgage holders cannot calculate the cash cost of these premiums, the risk of being misled appears high. [DES1]

The issue of "fairness"

Finally, we need to consider whether this mortgage offer is "fair". We note that the redemption fee is equivalent to the stated value of the cash back, yet the cash back is not worth nearly as much as the stated amount. In effect, the financial institution demands repayment of the put premium as well as the true value of the cash back should the mortgage redeem within the locked-in period.

To assess whether this is fair, consider the situation in the wholesale market. In the City option traders do not have the choice to renege on an option contract they have sold, for that would be a violation of a contract. They are truly locked-in; the only way they can get out of a short option contract is to buy back the contract. If we re-examine Figure 6 (Section 7.0) we can see that the value of the short option contract may be negative (at interest rate levels below B.E.). In this case, they will be buying back the option contract at a price above the price they had originally sold the contract, thereby realising a loss.

Should this "mark to market" rule apply to the financially unsophisticated mortgage holder? We think not for two reasons. First, how can the financially unsophisticated mortgage holder be sure they are being treated fairly? Unlike the City options market, there is no active secondary market in home mortgage interest rate floors, so they cannot rely on an efficient market to tell them what the fair market price of these options are. Since they are financially unsophisticated, they cannot replicate the price of the option independent of an active, efficient secondary market. They are dependent upon the financial institution to give them an accurate fair price. The obvious asymmetric information (financial institutions know more than the mortgage holder) problem can lead to abuse or, almost as bad, the perception of abuse. In summary, the mortgage holder is too vulnerable.

Second, mark to market redemption fees may create a negative externality. It is entirely possible that due to unfavourable (from the mortgage holder's perspective) variable rate movements, the cash cost of mark to market early redemption may be prohibitive. Remember, mortgage holders do not have the capital capacity of City brokerage firms so modest losses may indeed force them to continue with unfavourable mortgage deals. They may prevent them from moving, thus harming the mobility of labour.

On the other hand, the financial institution did pay the option premium, so they should at least be able to recover the cost. Hence it seems that demanding repayment of the true mortgage incentive plus the put premium is a reasonable early redemption fee.

In summary the value of this mortgage offer according to our estimates can be seen as:

Mortgage Advance + Cash Back = Straight Variable Rate Mortgage + Put Option

£157,000.00 + £9,670.00 > £157,839.69 + £5,474.98 = + £3,355.33.

In other words, the mortgage offer yields a £3,355 incentive for the mortgage holder.

7.3 Discounted Mortgages

Our third example is a discounted interest variable rate mortgage. The mortgage was offered on 21 February 1996. There were three separate elements, the first, a portion of the mortgage was an ordinary variable rate mortgage, and a second portion was a variable rate discounted mortgage and a third portion was a discounted endowment mortgage. We chose to examine the second portion, as it is a clear example of a variable rate discounted mortgage.

The relevant details are as follows. The discount was 3% (300 basis points) below the unrestricted variable rate on a mortgage amount of £15,000. The discount period ran from 21 February 1996 to 31 May 1997, slightly more than 15 months in all. The mortgage holder was tied in till 31 May 2001. Repayment within that time would incur a penalty calculated as Penalty = Amt x N x £2.5. Amt refers to the number of complete thousands loaned at the discount rate, N is the number of months the discount was enjoyed by the mortgage holder, and £2.5 was simply a multiplier that converted the previous terms into the corresponding cash penalty. Partial redemption would attract a pro-rata penalty since in that case Amt would be based on the number of complete thousands repaid. A final wrinkle is that at no time will the discount result in a mortgage rate below 1%. Hence, the mortgage holder has sold a put option since the mortgage rate is floored at 1%. Therefore our challenge is first, to value a 63-month (actually a few days more) variable rate mortgage, and second to value the 3% discount and to value a floor at 1%.

As before, we will value the package of securities included in this mortgage deal by breaking the deal down into its component securities. In particular, we represent the package as,

Mortgage Advance + (Discount-Floor) = Straight Mortgage + Redemption Fee.

As in example two, it is not necessary that the discount less floor put option value equals the redemption fee, or that the stated mortgage value (£15,000) equals the straight mortgage value. For example, the financial institution can implicitly claw back part of the value of the net discount value through under-advancing the true value of the underlying straight variable rate mortgage. Nevertheless the values obtained should make the above equation hold. Hence our methodology is first to value the unrestricted variable rate mortgage (straight mortgage), and second to value the discount and floor together and third to value the redemption fee.

We proceed in the usual way to find the value of a variable rate mortgage. However, in contrast to example two, this mortgage was agreed on 21 February of the year, when the bank's year-end is 31 December of the year. Like example one, we have to make a first year reconciliation between the day's interest charged and the monthly payments made[9].

We have to make two adjustments since we use the balance owing to calculate the payments to be made and of course adding the additional reconciliation amount will increase the final path dependent amount owing on 31 May 2001. These adjustments are easily made. To calculate the payments to be made, we simply adjust our usual formula by increasing the original repayment principal by the amount of the reconciliation (Add) from the date of reconciliation (December 1996) onwards. That is,

$$\frac{15{,}000 + \text{Add}}{\left[\dfrac{1-(1+r_{i,t})^{-T}}{r_{i,t}}\right]} \div 12$$

For the final principal amount, we again add the additional amount created by reconciliation to the principal outstanding in December 1996. We use this adjusted balance as the amount that is repaid for the remaining term. To see exactly what is involved, reconsider how we calculate the path dependent amount owing on a variable rate mortgage in state S (11,10) on 31 January 1997 one month after the reconciliation amount has been added to the amount owing. The following formula is the same one discussed in Section 7.1, only now we adjust the amounts owing by the reconciliation amount last period, either £73.19 in last period's low interest rate state, or £81.55 from last period's high interest rate state.

[9] We have to make two adjustments to our spreadsheet since we use the balance owing to calculate the payments to be made and of course adding the additional reconciliation amount will increase the final path dependent amount owing on 31 May 2001. See the spreadsheet for details (see page 2 for details of how to obtain it). You will find that in the principal section from column M onwards an additional amount is included. In the pricing section you will find that from column M onwards adding the reconciliation amount increases the monthly payment.

0.5*[(14819.29+73.19)-((14819.29+73.19)/((1-(1+4.870484/100)^-25)/((4.870484/100)/12))-((14819.29+73.19)*(1+4.870484/100*1/12)- (14819.29+ 73.19)))+ (14827.34+81.55**)-((14827.34+81.55)/((1-(1+4.870484/100)^-25)/((4.870484/100)/12))-((14827.34+81.55)*(1+4.870484/100*1/12)- (14827.34+81.55)))] = 14796.97**

Again note that the last term in bold represents the second mortgage calculation that makes up the average to account for path dependence. Noting this, we see that we add together the end of period amount owing if the beginning mortgage value was £14819.29+73.19 and the end of period amount owing if the beginning amount owing was £14827.34+81.55 and then taking an equally weighted average. This is the way we calculate the mortgage amount owing for all the “middle” states as we move forward in time. For all “extreme” states, that is, the very highest and lowest interest states at every time period, we simply use last period’s corresponding extreme amount owing plus the reconciliation as the beginning of period amount. As before, the end result of this process is a column of 64 possible expected amounts owing on 31 May 2001.

With the above adjustments, we see that our backward solving procedure finds the value of the underlying straight mortgage is £15,114.21. This amount is only slightly higher than the amount advanced, so it appears that a modest implicit administrative fee/incentive claw back is included in the mortgage offer.

Now we turn our attention to valuing the discount interest incentive. The first task is to value the incentive from the mortgage holder’s perspective. We chose to use a two-step process to simplify the programming. The first step is to work out the monthly cost savings due to the discount. We include the impact of the interest rate floor at 1% in this step. We then find the present value of this sequence of monthly savings. The second task is to find the present value of the cost of early redemption. We can then compare the incentive value to the redemption cost. If the mortgage offer were “fair” we would expect that the redemption fee is approximately equal to the value of the incentive, plus perhaps a modest administrative fee. As always, we would not expect that the redemption fee is much larger than the incentive because no early redemption inspired call option feature is present in a variable rate mortgage.

To work out the month-by-month value of the incentive, we simply multiply the beginning of period principal by the annual discount divided by 12[10]. We must divide by 12 to convert the annual discount to its monthly equivalent. However, the annual discount is not always 3%. This is true because if our interest rate tree generates an interest rate level that is below 4%, then the 1% floor becomes binding. Recall that the mortgage offer required that the mortgage holder continue to pay at least 1% interest after the discount. This means that when the unrestricted variable rate is below 4%, the mortgage holder receives a discount that is less than 3%.

[10] Note that the beginning of period principal includes the impact of the interest rate reconciliation from 21 January 1997 onwards.

To include the floor restriction, we calculate the annual discount rate using three rules at each possible interest rate level for the discount period. If the unrestricted variable rate is 4% or more, the annual discount is 3%. If the unrestricted variable rate is less than 4% but greater than 1%, then the annual discount is the unrestricted interest rate (say 3.5%) less 1% (so a discount of 2.5% is obtained). If the unrestricted interest rate is 1% or less then the discount would be zero[11]. Using these rules, we obtain the correct annual discount which when divided by 12 and multiplied by the beginning of period mortgage balance, obtains the monthly discount benefit. Note that this includes the interest rate floor provision included in the discount mortgage incentive.

The second step is to present value the sequence of monthly discount benefits. This is done using the usual backward solving procedure. The only wrinkle is that payments are made on the 21st of each month, but the discount period ends on 31 May 1997. Since the discount applies until 31 May, we find the monthly discount that is applicable if the incentive ran until 21 June 1997 and then discount this value back 21 days. This calculation represents the final monthly benefit that we use as our end point for our backward solving procedure. When we complete this procedure, we find that the value of the incentive is £568.09.

How does the early redemption fee compare to the value of the incentive? We note that the maximum amount to be charged is chargeable from 31 May 1997. According to the mortgage offer, the redemption fee payable at that date is Penalty = Amt x N x £2.5, so in this case the fee is 15 x 15 x 2.5 = £562.5. We can present value this amount back under the assumption of stochastic interest rates using our now familiar backward solving procedure. Note that this procedure finds the maximum value of the redemption fee, for delay of early redemption at later months will incur the same nominal cost (£562.50), but the delay will allow for more discounting to reduce the present value. When we carry out the backward solving procedure using 31 May 1997 as the end point we find that the redemption penalty is worth £512.67.

In total the mortgage offer equation is

Mortgage + (Discount-Floor) = Straight Mortgage + Redemption Fee

£15,000.00 + £568.09 < £15,114.21 + £512.67 = -£58.79.

which we see is roughly in balance. In contrast to example two, it is erroneous to conclude that the above inequality suggests that, rather than offering an incentive, the mortgage offer yields a net implicit administrative fee of £51.79. Note that the redemption penalty is not a value of a security sold at the onset of the mortgage and is incurred "voluntarily" if the mortgage holder chooses to redeem early. In example two, the mortgage holder has sold a put option at the onset of the mortgage offer, and has given up value equal to that amount regardless of whether they actually pay the cash value later through early redemption. In example three, the true value of the discount incentive is £568.09 less the implicit administrative fee £114.21 (£15,000 - £15,114.21) or £453.88.

[11] These rules are incorporated into the spreadsheet using 'if' statements. See page 2 for details of how to obtain a copy of the spreadsheet.

8.0 VARIABLE AND FIXED RATE DEPOSITS

The final issue examined in this report is whether the spread between fixed term and variable term deposits are "fair". We note that the choice of whether to lend to a bank at an instant access rate or at a fixed term rate involves more that just pecuniary costs and benefits. Instant access accounts offer a non-pecuniary "convenience yield". That is, instant access accounts offer safekeeping, record keeping and convenient transaction systems (checks, debit cards, electronic fund transfers). These features are valuable to the consumer and costly to provide. Hence the financial institution must charge for these services. In the UK, the tradition has been to charge implicitly for these functions by offering a low instant access savings interest rate and restrict the number of transactions per month rather than charge a cash fee.

On the other hand fixed term deposit accounts restrict access to the funds deposited for the contracted term. While they still provide safekeeping, they provide much less of the transaction system and record keeping benefits. Since these accounts are less costly, the financial institution offers higher interest rates.

In any event, both types of accounts represent valuable sources of financing to the financial institution, so some interest rate is offered on both accounts. However, the spread between them is partly determined by the much larger and non-pecuniary convenience yield offered on the instant access account. This means that the consumer will always face the choice to lend at a lower instant access rate or a higher fixed term deposit interest rate when making the choice to lend at convenient variable or less convenient fixed terms. To what extent the spread between these interest rates are determined by these non-pecuniary benefit/costs is difficult to model since it is always difficult to relate non-pecuniary benefits to pecuniary value; such is the subject of marketing rather than finance. However, we can analyse the spread in terms of its risk and then determine whether the spread reflects differential risk.

The problem is that both types of savings accounts are subject to some stochastic process. We note that an instant access account is less risky since the consumer can always withdraw and redeposit elsewhere if the account does not respond to a rise in interest rate rates. However, fixed term deposits are more risky since the consumer must wait till the term ends before they can re-lend elsewhere. So we ask, how can one determine whether the spread between variable and fixed term deposit rates reflect the differences in risk? If it does, then we will suggest that the spread is fair.

8.1 Modelling Fixed and Variable Term Spreads

We suggest that the choice to lend at fixed or variable terms is the same that faces a firm in the interest rate swap market. In the swap market, the firm may chose to convert a variable rate loan owed to a retail bank into a synthetic fixed rate loan in the following way. First, they agree to pay a fixed rate to a third party, usually an investment bank. In return, the investment bank pays a variable rate to the firm. The firm then uses the variable payment to cover the variable interest rate they pay to the retail bank. This eliminates the "variableness" of the retail variable rate loan and on a net basis they end up paying a fixed rate to the investment bank. This series of transactions with the investment bank is called a swap. When a firm thus converts a variable rate loan into a synthetic fixed rate loan, we say the firm buys a swap. On the other hand if they were to do the reverse and convert a fixed rate loan into a variable rate loan, we say the firm has sold a swap.

We can apply this swap structure to the choice to lend at fixed or variable terms in the consumer market. First, fixed term lending also implies *fixed rate* lending since there is no mechanism forcing the retail bank to adjust contracted interest rates during the fixed term. Second, variable term (instant access) lending also implies *variable rate* lending since if the retail bank fails to adjust instant access interest rates then the consumer can withdraw and re-lend elsewhere at competitive rates. This forces retail banks to respond to changes in interest rates and converts a variable term account into a variable interest rate account. Hence the choice to lend at fixed or variable terms easily converts into a choice to lend at fixed or variable rates.

The consumer's choice to *lend* at variable or fixed rates is the same as the firm's choice to *borrow* at fixed or variable rates. Hence we can value the choice to lend at variable or fixed rates in the same way we can value interest rate swaps. We know how to value a swap. When a consumer chooses to lend at fixed rates they are in effect buying a fixed rate deposit and selling a variable rate deposit. Note that the consumer does not actually sell a variable rate deposit but they do forgo the opportunity to do so. Hence we justify the short position in a variable rate deposit as an opportunity cost.

Therefore, the pricing equation of the initial choice to lend at fixed rather than at variable rates is,

$V_s = F - V.$

Note that F represents the value of the fixed rate deposit and V represents the opportunity cost of the forgone variable rate deposit. We see that when a consumer chooses to lend at fixed rate they in effect sell a swap. In the true swap market, the value of the swap will be zero at initiation because both parties (the firm and investment bank) are obligated for the full term of the contract.

However, there is an additional factor in the retail deposit market. When a consumer lends at variable rates they also bear less risk. This is true since if interest rates increase and the bank does not respond then they can immediately withdraw their funds and re-lend elsewhere at competitive rates. Hence the deposit always pays the going rate and is worth at least par. If they lend at fixed rates, they cannot do this because the deposit is actually fixed term. Consequently the value of the fixed term deposit may be worth less than face value if competitive fixed term lending rates increase.

Notice that this difference between lending at variable and fixed rates is not replicated in the true swap market since both the firm and investment bank are obligated to perform for the full term of the contract. In other words, once a firm agrees to pay a variable rate, they must continue to do so for the full time period of the swap. If they do decide to terminate the contract, then any losses/gains found using termination date interest rate scenarios have to be paid/received. To see this, consider the case where interest rates are now high when compared to the fixed rate. The firm must pay the difference between the value of the higher variable rate stream and the lower fixed rate stream. That is, the contract is marked to market. In contrast, the consumer is not obligated to continue to lend at variable rates to a particular bank. They can switch at any time to other alternatives with no mark to market adjustments. Hence we must adjust the above swap valuation equation to account for this.

This additional feature of the deposit market can be modelled as a put option. When a consumer lends at variable rates, they can put (sell back) the value of the deposit at face value should interest rates increase. In contrast, the fixed rate deposit does not have this feature. The retail deposit market will be in equilibrium only when consumers are indifferent whether they lend at fixed or variable rates. Therefore the fixed deposit rates must be increased to compensate consumers for the forgone put option should they choose to lend at fixed rates. This means our pricing equation is,

$V_s = F - V - P = 0.$

In other words, fixed deposit rates increase to reflect the value of the put option. In equilibrium the investor is indifferent between lending at fixed rate rates and obtaining a value F and forgoing the opportunity to lend at putable variable rates and obtain a value of V plus P. If the above equality holds, then we can say that the spread between variable and fixed deposit rates is fair since the fixed rate reflects the opportunity cost of the put option implicit in the ability to lend at alternative variable rates. If the above equality does not hold, then we can say that the spread is unfair. In the latter case we cannot say which interest rate should be changed since we can examine only the spread but not the level of each interest rate. As stated earlier, levels of interest rates reflect non-pecuniary costs and benefits that we do not examine.

8.2 Measuring Fixed and Variable Rate Deposit Yield Curves

Both fixed and variable deposit rates are subject to some stochastic process. Consistent with the methodology of the report, we will use Black Derman and Toy (1990) to model stochastic deposit rates. This means, like mortgage interest rates, we must calibrate the deposit interest rate process to an existing yield curve. The new problem we now face is that the consumer faces two yield curves, a variable rate and a fixed rate yield curve[12]. This happens because a spread will always exist between the two rates due to the different non-pecuniary costs and benefits offered by each.

However, it is not necessary to model the variable deposit rate yield curve since we know the value of the variable rate deposit is its face value (ignoring accrued interest) at each date the variable rate is changed. This happens because at the reset date the variable rate is changed to the competitive rate. Since today is a reset date, we know the variable rate is the discount rate that reflects opportunity cost and the interest rate that reflects the rate at which earnings accrue. As we discount and add future cash flows at the same interest rate, the value of the deposit is its face value.

[12] Note that the fixed deposit rate is still subject to a stochastic process since the offered fixed rate changes from time to time. In other words, the rate is fixed for a particular deposit, but the offered fixed rate to new customers changes as time continues.

The existing fixed rate yield curve must be estimated since in general the value of the fixed term deposit does not reset to any particular value as the fixed rate changes. To study how we are to do this, we first examine the correlation between 60- and 90- day deposit rates and T-bill rates. This information is presented in Tables 4 and 5.

Table 4
Correlation among 60-Day Deposit Rates and Treasury Interest Rates

	Time Deposits - 60-day					Treasury		
	Base	A&L	Barclays	Halifax	Midlands	RBS	1 Mo T-B	3 Mo T-B
Base	1	0.697531	0.931805	0.904828	0.856043	0.800295	0.968722	0.956565
A&L	0.697531	1	0.592484	0.620078	0.627691	0.617511	0.736345	0.741477
Barclays	0.931805	0.592484	1	0.956379	0.932744	0.860387	0.858751	0.825727
Halifax	0.904828	0.620078	0.956379	1	0.937158	0.901783	0.853808	0.827012
Midlands	0.856043	0.627691	0.932744	0.937158	1	0.942673	0.797262	0.752558
RBS	0.800295	0.617511	0.860387	0.901783	0.942673	1	0.753738	0.71445
1 Mo T-B	0.968722	0.736345	0.858751	0.853808	0.797262	0.753738	1	0.985493
3 Mo T-B	0.956565	0.741477	0.825727	0.827012	0.752558	0.71445	0.985493	1

Table 5
Correlation among 90-Day Deposit Rates and Treasury Interest Rates

	Time Deposits-90 day							
Base	Abbey	C&G	Wool	TSB	Cov	Yorks	1 Mo T-B	3 Mo T-B
Abbey	1	0.923858	0.949365	0.901231	0.944798	0.841529	0.892105	0.968722
C&G	0.923858	1	0.944496	0.831767	0.946819	0.770652	0.904985	0.896925
Wool	0.949365	0.944496	1	0.912628	0.946306	0.863792	0.918301	0.9227
TSB	0.901231	0.831767	0.912628	1	0.902499	0.889375	0.872589	0.839744
Cov	0.944798	0.946819	0.946306	0.902499	1	0.794552	0.911674	0.908098
Yorks	0.841529	0.770652	0.863792	0.889375	0.794552	1	0.832435	0.758854
1 Mo T-B	0.892105	0.904985	0.918301	0.872589	0.911674	0.832435	1	0.828847
3 Mo T-B	0.968722	0.896925	0.9227	0.839744	0.908098	0.758854	0.828847	1

Correlations shown in Tables 4 and 5 have been tabulated from deposit rates provided by the Bank of England and T-Bill rates from *Datastream*. These monthly series cover the 31 January 1995 to 30 April 1999 time period. Table 4 shows that with only one exception (A&L) 60-day time deposit rates are more highly correlated with one-month T-bill rates than three-month T-bill rates. Table 5 shows that with two exceptions (Abbey National and Woolwich) 90-day time deposit rates are more highly correlated with one-month T-bill rates than three-month T-bill rates.

Consequently we measure the existing 60-day deposit yield curve as the existing Treasury yield curve less the spread between the one-month T-Bill yield and the 60-day deposit rate. The spread is subtracted at all points along the Treasury yield curve to obtain the 60-day deposit yield curve since 60-day deposit rates are always below the corresponding one-month T-bill rate. 60-day deposit rates are always lower because the deposit rate reflects non-pecuniary benefits associated with safekeeping and record keeping. For the 90-day yield curve we follow a similar strategy only we choose to subtract the spread between three-month T-bill rates and 90-day deposit rates. We choose to use this spread in this case since our 90-day time deposit example is based on Abbey National rates. For Abbey National, 90-day deposit rates are more highly correlated with three-month T-bill rates.

8.3 Deposit Spread Examples

We conduct two empirical examples of the deposit spread. The first is spread between Barclays' 60-day time deposit rate and instant access rate on 31 December 1998. The time deposit rate was 5.05% and the instant access rate was 2.75% implying a 230-basis point spread. The second is the spread between Abbey Nationals' 90 day time deposit rate and instant access rate on 31 December 1998. The time deposit rate was 4.72% and the instant access rate was 1.4% implying a 332-basis point spread. These spreads should be attributable to a convenience yield due to the non-pecuniary benefits derived by consumers and the opportunity cost of the lost put option should the consumer choose to deposit for a fixed rather than a variable term.

Since we are dealing with very short-term instruments, we adjust our Treasury yield curve from monthly to daily interpolation. This is easy to do since we have estimated the Treasury yield curve via Nelson and Siegel (1987) which provides a discount function that is used to estimate the yield curve. To convert a monthly interpolated yield curve into a daily interpolated yield curve, all we have to do is to adjust the time parameter included in Nelson and Siegel's (1987) discount function from 1/12 (monthly) to 1/365 (daily).

For each example we proceed in the usual way by first modelling the stochastic fixed term deposit rate process using Black Derman and Toy (1990). Our short-term interest rate is the one-month T-bill rate less the spread between the one-month T-bill rate and 60-day deposit rate for Barclays. For Abbey National, our short-term interest rate is the one-month T-bill rate less the spread between the three-month T-bill rate and the 90-day deposit[13]. Volatility is 23.5% on 31 December 1998.

For both examples we find the value of the fixed rate deposit and the implied put by solving backwards in the usual way. We assume monthly compounding of interest. The results of this process are summarised in Table 6.

Table 6
Fixed vs Variable Rate Results-31 December 1998

	Barclays Bank (60-Day)	Abbey National (90-Day)
Time Deposit Rate	5.05%	4.72%
Instant Access	2.75%	1.4%
Spread (in Basis Points)	230	332
Value of Time Deposit	£100,045.89	£100,034.06
Less-Value of Instant Access Account	£100,000	£100,000
Less-Value of Put	£44.00	£294.63
Difference	£1.89	-£260.57

We find that Barclays' spread closely matches our model. The small observed difference in value suggests that the spread is too wide by less than one half of a basis point. In contrast, Abbey National's spread appears too narrow since a large negative

[13] The one-month interest rate is the shortest Treasury interest rate that is available.

value is observed. In fact, the spread should be wider by 21 basis points (0.21%) if the value of the time deposit is to equal the sum of the value of the instant access account and put.

Why does Abbey National compare so unfavourably with Barclays? We noted from Tables 5 and 6 that most 90- and 60-day time deposit rates correlate with one-month rather than three-month T-bill rates. Abbey National was one of only three exceptions, so we modelled Abbey National's lending curve using the three-month T-bill less the 90-day deposit rate as the spread deducted from the Treasury yield curve. On this day, the Treasury yield curve was downward sloping at the short end. This means that the spread deducted from the Treasury yield curve was smaller, and a higher lending curve resulted. With a higher lending curve, the value of the fixed rate deposit will be less than par value since more discounting will occur at higher lending rates. This will create more "in the money" states for the put and increase its value.

Since this might make a difference we replicated our results using a lower lending curve derived by deducting the one-month less 90-day deposit spread from the Treasury yield curve. In this scenario we find that the above *negative* £260.57 difference changes to a *positive* £55.62. This implies that rather than being too narrow, the spread between fixed and variable deposit rates should increase modestly by nine basis points. One again (see Section 7.2) this highlights the fact that the value of a derivative is sensitive to the value of the inputs. One suspects that a better fit to our model may be obtained by using the spread between the two-month T-bill and 90-day time deposit rates. However a two-month T-bill rate was unavailable from *Datastream.*

9.0 POLICY IMPLICATIONS

We believe there is an objective standard for assessing whether a redemption charge is fair. To apply this standard it is necessary to first break down the mortgage into its component securities, value each one separately and sum the resulting values. The value thereby obtained should equal the stated cost of the mortgage plus the redemption fee, plus at most a modest fee to recover the incremental costs incurred in administering the redemption.

We believe that it is practical to apply this standard. We suggest a theoretically correct methodology for valuing a mortgage and its component securities. We conduct numerical examples showing that practical difficulties can be overcome. We note that the valuation methods suggested in this report are of the same type as valuation methods used in the City to trade securities in the global capital markets. We are therefore confident that the methodology we have proposed (or a variation thereof) could be routinely applied by the Office of Fair Trading when assessing whether terms specifying early redemption charges in mortgage contracts are "unfair" according to the Unfair Terms in Consumer Contracts Regulations 1999.

In at least one instance we also found that at least part of the stated value of an incentive actually represented the value of a security sold by the mortgage holder. In such instances, the stated value of the incentive may be misleading. The stated value implies that this is the amount the financial institution is willing to pay to obtain the mortgage holder's business when in fact a large fraction of the incentive can actually represent a payment to the mortgage holder for accepting an obligation.

[DES2]

[RM3]

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APPENDIX A1

The purpose of this appendix is to explain the construction of the Excel spreadsheet program that is used to value fixed rate mortgages and the associated call option. We also explain how one is to use this model. This appendix is organised as follows. First, we describe the Excel spreadsheet layout of Black Derman and Troy (1990), constant volatility version. It is hoped that this description, along with the attached spreadsheet, will be invaluable in programming the model more efficiently using more sophisticated programming software. Second, we describe how the model is calibrated to the existing term structure using the solver utility found in the “tools” heading on the Excel toolbar. Finally, we describe how we operationalize the pricing methodology to price a fixed rate mortgage and the associated call option.

A1.1 Description of the Spreadsheet Layout

The first three rows show some summary statistics of the model, the short-term lending rate (one-month T-bill yield plus the lending spread of 149 basis points), time step (one month or 1/12 of a year), date and volatility (expressed as a standard deviation). Row 5 represents the calibration factors u_t.

Starting in cell A71 we have our interest rate tree. Cell A71 is the current one-month interest rate (see cell B2), and cells B70 and B71 represent the up and down short-term interest rates next month as found by our interest rate model. We can see this if we recognise that the formula in the cell *is* the Black Derman and Toy (1990) model (constant volatility version). Look at cell B70, and we can realise that the Excel formula,

=A71*EXP((SUM(A5)*D2+(2*B6-B6)*B3*D2^0.5)/100)

is equal to the up interest rate state formula,

$$R_{t,i} = R_{0,0} x \exp\left[\sum_{t}^{N} u_t T + \sigma\sqrt{T}\right]$$

That is, $R_{0,0}$ is A71, Σu_t is SUM (A5) since row five are our annual calibration factors, and T is D2, our time step that converts the annual calibration factor to its monthly equivalent. The next part of the cell (2*B6-B6) gives the number of up ticks in interest rates in period one (the second period because the first period is noted as 0). Note that B6 is 1, so this expression gives the value of 1 or one uptick above the lowest possible state. This value is multiplied by the standard deviation (volatility) B3, which in turn is multiplied by the square root of time D2^2. This corresponds to adding one uptick in interest rates multiplied by the square root of time to convert an annual volatility uptick to its monthly equivalent, as shown in the Black Derman and Toy (1990) model[14].

[14] Why multiply by the square root of time? Because volatility is a standard deviation, so it is measured in standard deviation units. To convert an annual number in standard deviation units to a monthly number, you multiply by time (expressed as a fraction of a year) also measured in standard deviation units.

Cell B71 which represents the value of the new short rate next period should interest rates decline is exactly the same as Cell B70, except than the tick portion of the spreadsheet formula is (2*A6-B6). Since A6 is zero this expression yields a value of –1, or one downtick in interest rates. Looking further out in the interest rate tree you will observe that this formula is always the same, except that the number of upticks and downticks changes corresponding to the interest rate level you are examining, and the calibration factors are summed for all prior and current period values.

Starting at cell A134, we have our discount factor tree. This tree represents the one period discount factors that value the receipt of £1 at the end of the period to its value at the beginning of the period. Looking at cell A134, we see that the spreadsheet formula is

=EXP ((-A71/100)*D2).

This is equivalent to $Exp^{-(R_{t,i}T)}$. Here A71/100 is $R_{0,0}$ or the current one-month lending rate in decimal value. We multiply this rate by the time step of one-month D2 to convert the discount factor to a monthly figure. Looking further out in the tree we observe that all formulas in the discount factor tree are exactly the same except that relevant interest rate changes to correspond to the relevant short-term lending rate at that time and state according to our interest rate tree.

Starting from cell A203 we have our Arrow-Debreu security price tree. Looking at cell C202 we see that the Excel formula

=B201*0.5*B133+B202*0.5*B134

is

$$A(t,i) = .5x[A(t-1,i)xd(t-1,i) + A(t-1,i-1)xd(t-1,i-1)]$$

Note that B201 represents A (2,1), B133 is d (2,1), B (202) is A (2,0) and B (134) is d (2,0). Therefore C (202) is A (2,1), the "middle" Arrow-Debreu price in period two. This security represents the value today of a security that pays £1 for sure *if* interest rates reach the middle interest rate state in period two, and pays zero otherwise.

Row 205 is the addition of columns of Arrow-Debreu prices. These cells represent our replicating portfolio, being the purchase price of one Arrow-Debreu security each for all possible interest rate levels that may evolve in the relevant time period. Since this replicating portfolio will pay-off £1 for sure no matter what interest rate level arrives in the relevant period, this replicating portfolio replicates the pay-off and risk of investing in a Gilt zero.

Row 206 finds the zero yield of the replicating portfolio. For example, look at cell B (206), which is =-LN (B205)/(D2*B208)*100. This cell represents the annual yield on the two-month replicating portfolio, which should be equal to a two-month Gilt yield. We take the negative of the natural logarithm of the decimal price, which gives the total return. We then divide by the time step of one month multiplied by two. In other words, we divide by 1/6 (or multiply a two-month return by 6), which converts the total return into an annual return.

Row 209 is the zero coupon yield as estimated by Nelson and Siegel (1987) for 14 August 1996. Therefore row 206 should equal row 209 in order to prevent pure arbitrage.

A1.2 Calibration of the Model

We wish to calibrate 58 different columns of interest rates, discount factors and Arrow-Debreu prices so that the addition of columns of Arrow-Debreu prices (row 206) obtains a yield that agrees with the zero coupon spot curve (row 209). The following is an explanation of how to do this in one step, rather than 58 different steps![15]

Under the tools menu on the toolbar, click solver. Your target cell is B71. This is the lowest interest rate level that arrives at the beginning of period two. For the next item, chose the target cell to equal a maximum. Then in the change cells, click on the red arrow. This will give you a one-line window. Move to cell A5, and click on it. This is the first calibration factor that calibrates the first interest rate, discount factor and Arrow-Debreu column such that they all agree with the two-period spot yield curve. Then press and hold shift, press end, and then press the right arrow on the keyboard. In other words, shift + end, shift + right arrow. This will fill in the target cell with a row from A5 to BJ5. Since the example is only 58 periods (not 61) delete the “J” and replace it with “G”. The end result, is that you have a row of calibration factors from A5 to BG5 as you change cells. Now click on the red arrow again to get back to the solver menu.

Now we wish to add a constraint. Click on Add. You will see a new window. Click on the red arrow for cell reference. Then click on cell B206 (NOT A206 since the first interest rate is automatically calibrated, the short spot rate is interest rate risk free). Then holding shift, press end, then the right arrow (like above). This adds a row from B206 to BI206. Again, since we are calibrating 58 months, not 61, we replace the “I” with an “H”. Then click the arrow in the next section of the “Add Constraint” window, and select equal to (=). Now click the last red arrow, and click on B209. Holding shift, press end, then the right arrow on the keyboard. This should fill in the constraint row from B209 to BH209. Then click OK.

You are now ready to run the solver program. However, you may wish to change the solver options as shown under the “options” selection in the main solver menu. These options control the accuracy of the results. Generally the tighter the precision, tolerance and conversion, the slower will be the program. If you ask for extremely tight values, you may find the program will not even converge! However, the default settings are inaccurate, so I have changed the settings to obtain extremely accurate results without problems. If you have changed the options, click on OK so you can get

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[15] We need to calibrate one more interest rate than the maturity of the underlying mortgage for technical reasons. In particular, we need the beginning of period 58 interest rate in order to measure the 57th period Arrow-Debreu security price.

back to the main solver menu. Now click on solve, and fetch yourself a cup of tea. When the program is finished, it will tell you that everything is fine, so click on OK.

A1.3 Finding the Mortgage and Implied Call Option Values

At cell A216 we have our mortgage cash flow. There are a number of cells around there that give some summary statistics of the mortgage under examination, the quoted fixed rate, the IRR (non-truncated version of the APR), the mortgage amount and so on. The important cells are rows 226, 227 and 228, which show the monthly mortgage payment, interest portion thereof and redemption charge.

For the first year (prior to reconciliation at the end of March for this building society), the mortgage payment is calculated in cell B217, so this value fills in the first few cells of row 226. The corresponding interest portion of this mortgage payment in row 227 is constant for the first few months since the building society credits principal repayments only at the end of the year. The interest portion is calculated as the amount borrowed at the beginning of the year (cell D216) plus one month's interest less the amount outstanding at the beginning of the year. The redemption charge (row 228) is simply the number of months' interest penalty times the monthly interest (row 227).

At the building society's year-end, the end of March in this case, the building society will reconcile the number of payments made (seven) with the day's interest to be charged. You will recall that in Section 4.3 we discussed that this is necessary since the building society calculates monthly payments based on an annual annuity assuming 12 monthly payments but in the first year less than 12 payments are actually paid. Cell G217 calculates this reconciliation, adding this amount to the original amount borrowed to recalculate a higher mortgage payment from April 1997 onwards.

Also, from April 1997, the interest portion is reduced because the repayments of principal made during the year reduce the outstanding principal. This new principal amount is calculated in cell H219 and is used as the basis for the new interest portion of the monthly payments in row 227, from Cell I227 onwards. This is done in exactly the same manner as above, only we use the lower balance outstanding (cell I227) rather than the original amount (cell D216). This of course reduces the redemption penalty. Each year in April, a new lower principal amount is calculated in row 219 to be used as the next year's lower interest portion of the mortgage payment in row 227.

Now that we have calculated the mortgage cash flows, we are able to calculate the value of the mortgage and the call option. Cell B288 shows the value of the mortgage when the lending curve is used. This value is found by the backward solving method as described in Section 2.25 (and Figure 5).

If we look at cell BE233, we find the value of the mortgage in 57 periods as of 14 April 2001. First we find the final principal amount remaining as of 30 April 2001 plus the last "coupon" or mortgage payment. The only new item here is that mortgage payments are made on the 14th of the month, but the building society assumes payments are received at the end of the month. We fix this by discounting the last cash flow back an extra 17 days. We do the same for all interest rate levels as of 14 April 2001, 58 cells in all. Then we apply the backward solving routine of Section

2.25, always taking 50% of the values of adjacent next period values, adding the mortgage payment (coupon), and discounting back one period using same state discount factors from our (calibrated) discount factor tree. The final value of the mortgage is £80,864.04 as reported in cell B288.

Note that rows 292 to 294, columns A to D show the summary values of the mortgage and redemption (call) features from the bank's and mortgage holder's perspectives.

The next tree starts at cell C353. This is our intrinsic value call tree. It simply calculates the value of the cost-saving annuity implied by the difference between the current variable rate as generated by our (calibrated) interest rate tree and the stated rate of 7.85%. The procedures used to obtain these values are described in Section 5.1. At this point, we do not worry if the intrinsic value is positive or negative.

The final tree starts at cell C411. This is the American value of the call option implicitly given to fixed rate mortgage holders. We find this value according to the procedures as described in Section 5.2. Namely, we first accept that the terminal values of the call feature (at period 56) is the American value since the American option is at the maturity date at that point in time. Then we compare the intrinsic value (found as the intrinsic value in the intrinsic value tree if positive, zero otherwise) to the American value, the value obtained if we delay exercise by one period for every node of the remaining 55 periods. We always take the larger of the two. Note that programming-wise, this is very easy to do using the logic operator IF statement in Excel.

APPENDIX 2

There follows:

Figure A2.1: Base Rate Levels

Figure A2.2: One Year Interest Rate Cap Volatility

Appendix A2:Base Rate Levels

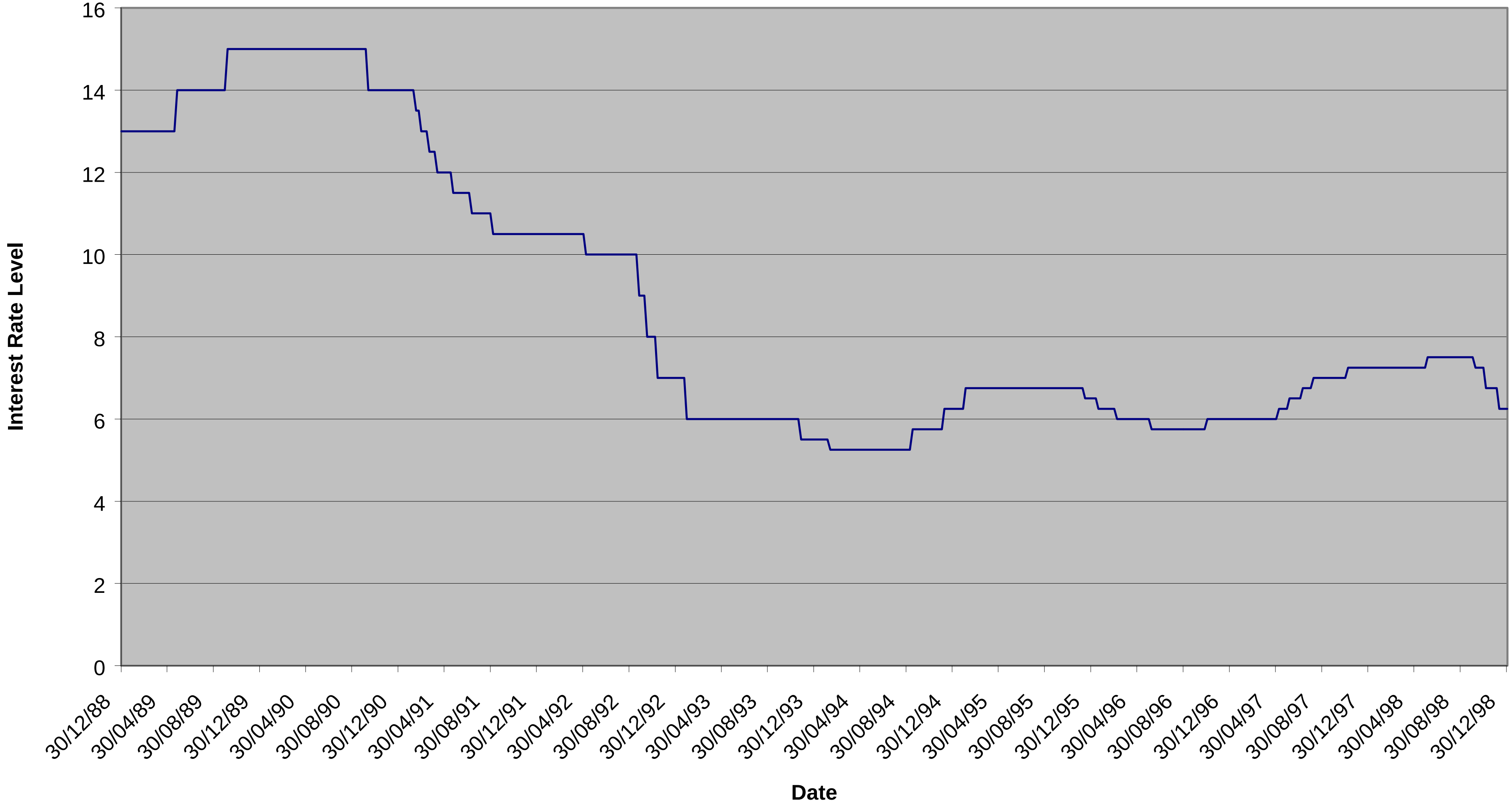

Figure A2: One Year Interest Rate Cap Volatility